

SUBCOMPLETIONS OF REPRESENTABLE RELATION ALGEBRAS

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ABSTRACT. Many finite symmetric integral non-representable relation algebras, including almost all Monk algebras, can be embedded in the completion of an atomic symmetric integral representable relation algebra whose finitely-generated subalgebras are finite.

1. Introduction

Monk [26] proved that if \mathbf{B} is a Boolean algebra with operators, then \mathbf{B} has a unique completion \mathbf{C} , where \mathbf{C} is a **completion** of \mathbf{B} if \mathbf{C} is a complete, \mathbf{B} is a subalgebra of \mathbf{C} , and \mathbf{B} is **dense** in \mathbf{C} , which means that below every non-zero element of \mathbf{C} there is a non-zero element of \mathbf{B} . Monk proved that if \mathbf{B} is a relation algebra algebra, then its completion is also relation algebra. The problem remained, if \mathbf{B} is representable, must its completion also be representable? In other words, is the variety¹ \mathbf{RRA} of representable relation algebras closed under completions? Hodkinson [15] provided the answer that, no, \mathbf{RRA} is not closed under completions because there is an *atomic* $\mathbf{B} \in \mathbf{RRA}$ such that the completion of \mathbf{B} is not representable.²

Consider any atomic representable relation algebra $\mathbf{B} \in \mathbf{RRA}$ whose completion \mathbf{C} is not representable, as might arise from Hodkinson's proof. Since \mathbf{RRA} is a variety and \mathbf{C} is not in \mathbf{RRA} , there must be an equation ϵ that holds in \mathbf{RRA} but fails in \mathbf{C} . Let \mathbf{A} be the subalgebra of \mathbf{C} that is generated by the finitely many values assigned to the variables occurring in ϵ . Then \mathbf{A} is also not representable because it fails to satisfy ϵ . Thus \mathbf{A} is an example of a finitely-generated relation algebra which is a subalgebra of the non-representable completion of an atomic \mathbf{RRA} . The question addressed by this paper is, which relation algebras can occur as \mathbf{A} ? We rephrase this question as a problem.

Problem 1. *Let K be the class of finitely-generated subalgebras of non-representable completions of atomic representable relation algebras. Which relation algebras are in K ? Does K contain any relation algebras that are not weakly representable?*

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¹It is easy to show that \mathbf{RRA} is closed under subalgebras and direct products. Closure under homomorphic images was first proved by Tarski [33] using model theory; for a more direct proof see [24, Th. 121].

²For subsequent developments, alternate and simpler constructions, extensions, and related work, see, for example, Andr  ka-N  meti-Sayed Ahmed [2], Hirsch-Hodkinson [9, 10, 11, 12, 13], Hodkinson-Venema [16], Khaled-Sayed Ahmed [17, 18], Sayed Ahmed [27, 28, 29, 30, 31], and Sayed Ahmed-Samir [32].

Some finite symmetric integral relation algebras have no proper extensions at all and are therefore neither representable nor in K ; see Frias-Maddux [6] for examples.

As a partial positive answer, it will be shown in this paper that every finite Monk algebra with six or more colors is in K . Monk algebras are relation algebraic versions of cylindric algebras used by Monk [25] to prove that classes of finite-dimensional representable cylindric algebras are not finitely axiomatizable.

2. Monk algebras

Definition 1. For $4 \leq q \in \omega$, the **no 1-cycles algebra** \mathbf{E}_q^{23} is the finite symmetric integral relation algebra with q atoms $\mathbf{e}_0 = 1'$, $\mathbf{e}_1, \dots, \mathbf{e}_{q-1}$ such that if a, b are distinct diversity atoms then $a; b = 0'$ and $a; a = \bar{a}$.

These relation algebras were constructed in [21] and were called $\mathfrak{E}_q(\{2, 3\})$ in [23, Def. 2.4, Prob. 2.7]. Two particular examples of these algebras, namely \mathbf{E}_4^{23} and \mathbf{E}_5^{23} , are 62_{65} and 3009_{3013} , respectively, in [24]. It is likely that \mathbf{E}_q^{23} is representable for all $q \geq 4$. This is known to be true for $q = 4$; the earliest reference to the representability on a 13-element set of \mathbf{E}_4^{23} is in a footnote by Lyndon [19]. In fact, \mathbf{E}_4^{23} is isomorphic to a subalgebra of the complex algebra of the 13-element cycle group \mathbb{Z}_{13} , and \mathbf{E}_4^{23} has exactly two other square representations, both on a 16-element set. Furthermore, \mathbf{E}_q^{23} was shown to be representable by Comer [4] for $q = 5, 6$ and more recently (by Comer's method) for $q = 7, 8$.

\mathbf{E}_q^{23} can also be described by cycles and atom structures, which are defined for algebras in NA, the class of **non-associative relation algebras**. An axiom set for NA is obtained by deleting the associative law from the axioms for relation algebras; see [20, Def. 1.2] and [24, Th. 314]. The **atom structure** [20, Def. 3.2] of an algebra $\mathbf{A} \in \text{NA}$ is $\langle \text{At}(\mathbf{A}), C, \sim, I \rangle$ where $\text{At}(\mathbf{A})$ is the set of atoms of \mathbf{A} , C is the set of triples of atoms $\langle x, y, z \rangle$ such that $x; y \geq z$, \sim is the restriction of the converse operation of \mathbf{A} to the atoms of \mathbf{A} , and $I = \{x : 1' \geq x \in \text{At}(\mathbf{A})\}$. In every NA, C is the union of sets of the form

$$(1) \quad [x, y, z] = \{\langle x, y, z \rangle, \langle \check{x}, z, y \rangle, \langle y, \check{z}, \check{x} \rangle, \langle \check{y}, \check{x}, \check{z} \rangle, \langle \check{z}, x, \check{y} \rangle, \langle z, \check{y}, x \rangle\},$$

where $x, y, z \in \text{At}(\mathbf{A})$. Such sets are called **cycles**. If $1'$ is an atom of \mathbf{A} , then the cycle $[x, y, z]$ is said to be an **identity cycle** if $1' \in [x, y, z]$, and a **diversity cycle** otherwise. If \mathbf{A} is **symmetric**, i.e., $\check{x} = x$ for all x , then a diversity cycle $[x, y, z]$ is said to be a **1-cycle**, **2-cycle**, or **3-cycle** if the cardinality $|\{x, y, z\}|$ is 1, 2, or 3, respectively. For example, the cycles of \mathbf{E}_q^{23} are all the 2-cycles and 3-cycles, but none of the 1-cycles.

Definition 2 (Andréka-Maddux-Németi [1]). Let \mathbf{A} and \mathbf{B} be atomic relation algebras. We say that \mathbf{A} is obtained from \mathbf{B} by **splitting** if $\mathbf{B} \subseteq \mathbf{A}$, every atom x of \mathbf{A} is contained in an atom $c(x)$ of \mathbf{B} , called the **cover** of x , and for all $x, y \in \text{At} \mathbf{A}$, if $x, y \leq 0'$ then

$$(2) \quad x; y = \begin{cases} c(x); c(y) \cdot 0' & \text{if } x \neq \check{y} \\ c(x); c(y) & \text{if } x = \check{y}. \end{cases}$$

Definition 3 (Andréka-Maddux-Németi [1, Ex. 6]). A **Monk algebra** is an atomic symmetric integral relation algebra obtained by splitting from some \mathbf{E}_q^{23} , $4 \leq q \in \omega$.

Assume \mathbf{A} is a Monk algebra obtained from \mathbf{E}_q^{23} by splitting. Then \mathbf{A} extends \mathbf{E}_q^{23} and the $q - 1$ diversity atoms of the subalgebra $\mathbf{E}_q^{23} \subseteq \mathbf{A}$ are called the **colors**

of \mathbf{A} . Consider a subalgebra $\mathbf{E} \subseteq \mathbf{E}_q^{23} \subseteq \mathbf{A}$. Then the Monk algebra \mathbf{A} extends its subalgebra \mathbf{E} in a way that, for want of a better name, we simply call “special”.

Definition 4. If \mathbf{A} and \mathbf{E} are finite symmetric integral relation algebras, then \mathbf{A} is said to be a **special extension** of \mathbf{E} if $\mathbf{E} \subseteq \mathbf{A}$ and for all diversity atoms $0' \geq a, b, c \in \text{At}(\mathbf{E})$,

- (i) if not $(a = b = c)$ and $a; b \geq c$ then $x; y \geq c$ whenever $a \geq x \in \mathbf{At}(\mathbf{A})$ and $b \geq y \in \mathbf{At}(\mathbf{A})$,
- (ii) if $a; a \geq a$ then $x; y \cdot a \neq 0$ whenever $a \geq x, y \in \mathbf{At}(\mathbf{A})$.

Every finite symmetric integral relation algebra is a special extension of itself. Every finite symmetric integral relation algebra with no functional atoms³ is also a special extension of its minimum subalgebra, the one whose atoms are $1'$ and $0'$.

Lemma 1. Every Monk algebra obtained from \mathbf{E}_q^{23} by splitting is a special extension of every subalgebra of \mathbf{E}_q^{23} .

Proof. Assume $\mathbf{E}_q^{23} \subseteq \mathbf{A}$, $4 \leq q$, \mathbf{A} is a Monk algebra obtained from \mathbf{E}_q^{23} by splitting, and $c(x)$ is the atom of \mathbf{E}_q^{23} containing the atom x of \mathbf{A} . Consider a subalgebra $\mathbf{E} \subseteq \mathbf{E}_q^{23} \subseteq \mathbf{A}$ and diversity atoms $0' \geq a, b, c \in \text{At}(\mathbf{E})$.

To show part (i) of Def. 4, we assume not $(a = b = c)$, $a; b \geq c$, $a \geq x \in \text{At}(\mathbf{A})$, and $b \geq y \in \text{At}(\mathbf{A})$. We want to prove $x; y \geq c$, so we also assume $c \geq z \in \text{At}(\mathbf{A})$ and must now show $x; y \geq z$. Note that $x \leq c(x) \leq a$, $y \leq c(y) \leq b$, and $z \leq c(z) \leq c$. Also, not $(c(x) = c(y) = c(z))$ since otherwise we would have $a = b = c$, contradicting our assumption. Hence $[c(x), c(y), c(z)]$ is not a 1-cycle, and is either a 2-cycle or 3-cycle. Now \mathbf{E}_q^{23} contains all 2-cycles and 3-cycles by definition, so $[c(x), c(y), c(z)]$ is a cycle of \mathbf{E}_q^{23} , hence $c(x); c(y) \geq c(z)$. By definition of splitting, x and y have a product equal to the product of their covers in \mathbf{E}_q^{23} , so we have $x; y = c(x); c(y)$. If $c(x) \neq c(y)$ then $x; y = c(x); c(y) = 0' \geq c$ by Def. 1. If $c(x) = c(y)$ then $a = b$, hence $a = b \neq c$ by the assumption not $(a = b = c)$. Therefore $a \cdot c = 0$ since $a, c \in \text{At}(\mathbf{E})$. By Def. 1 we get $x; y = c(x); c(y) = c(x); c(x) = \overline{c(x)} \geq \bar{a} \geq c$.

To show part (ii) of Def. 4, we assume $a \geq x, y \in \text{At}(\mathbf{A})$ and $a; a \geq a$. We wish to show that $x; y \cdot a \neq 0$. Note that a cannot be an atom of \mathbf{E}_q^{23} because the assumption $a; a \geq a$ fails for all atoms of \mathbf{E}_q^{23} by Def. 1. Assume first that $c(x) = c(y) = u$. Since a is not an atom of \mathbf{E} , a is the join of two or more atoms of \mathbf{E}_q^{23} , hence there is some atom $v \in \text{At}(\mathbf{E}_q^{23})$ such that $u \neq v \leq a$. We have $x; y = c(x); c(y) = u; u$ by the definition of splitting, but $u; u = \bar{u} \geq v$ by Def. 1, so $0 \neq x; y \cdot v \leq x; y \cdot a$, as desired. Assume that $c(x) \neq c(y)$. Then $c(x); c(y) = 0'$ by Def. 1, so $x; y = c(x); c(y) = 0' \geq a$, hence $x; y \cdot a \neq 0$. \square

Lemma 1 suggests that we consider an arbitrary subalgebra \mathbf{E} of \mathbf{E}_q^{23} . Every subalgebra contains $1'$, but $1'$ is an atom in \mathbf{E}_q^{23} , so it is an atom in \mathbf{E} as well. Thus \mathbf{E} is integral, but \mathbf{E} is also symmetric since it is the subalgebra of a symmetric algebra. The diversity atoms of \mathbf{E} are disjoint and join up to $0'$, so they partition the diversity atoms of \mathbf{E}_q^{23} . In every relation algebra, the relative product $a; b$ of *distinct* diversity atoms a, b of \mathbf{E} is included in $0'$. On the other hand, in this case we have $a; b \geq 0'$ because there are atoms x, y of \mathbf{E}_q^{23} such that $a \geq x$, $b \geq y$, and $a; b \geq x; y = 0'$ by Def. 1. Every diversity atom of $a \in \mathbf{E}$ satisfies either $a; a = 1$

³A functional atom x in a symmetric integral relation algebra satisfies $x; x = 1'$.

or $a; a = \bar{a}$, for if a is an atom of \mathbf{E}_q^{23} (as well as \mathbf{E}) then $a; a = \bar{a}$ by Def. 1, while if a is not an atom of \mathbf{E}_q^{23} , then it is the join of two or more atoms of \mathbf{E}_q^{23} , say $a \geq \mathbf{e}_1 + \mathbf{e}_2$, so

$$\begin{aligned} a; a &\geq (\mathbf{e}_1 + \mathbf{e}_2); (\mathbf{e}_1 + \mathbf{e}_2) \\ &= \mathbf{e}_1; \mathbf{e}_1 + \mathbf{e}_1; \mathbf{e}_2 + \mathbf{e}_2; \mathbf{e}_1 + \mathbf{e}_2; \mathbf{e}_2 \\ &= \bar{\mathbf{e}}_1 + 0' + 0' + \bar{\mathbf{e}}_2 && \text{Def. 1} \\ &= 1. \end{aligned}$$

Every subalgebra of \mathbf{E}_q^{23} can therefore be characterized by just two parameters: α , the number of diversity atoms a satisfying $a; a = \bar{a}$, and β , the number of diversity atoms satisfying $a; a = 1$. The number of atoms in the subalgebra is $1 + \alpha + \beta$. The only restrictions on these parameters are $\alpha + 2\beta < q$ and $0 < \alpha + \beta$.

An atom a of a symmetric integral relation algebra is said to be **flexible** if $a; a = 1$ and $x; a = 0'$ for all diversity atoms x distinct from a . Having a flexible atom is a sufficient condition for representability; see Comer [5, 5.3] or [22, Th. 6]. Since every proper subalgebra of \mathbf{E}_q^{23} has at least one atom a satisfying $a; a = 1$ and this atom is flexible by Def. 1, every proper subalgebra of \mathbf{E}_q^{23} is representable.

3. An infinite atom structure from two finite algebras

In this section we use an arbitrary finite symmetric integral relation algebra \mathbf{A} and its subalgebra \mathbf{E} to construct a complete atomic algebra $C_{\mathbf{E}}(\mathbf{A}) \in \mathbf{NA}$ that has subalgebras isomorphic to \mathbf{A} and \mathbf{E} .

The **complex algebra** of the structure $\langle A, C, \smile, I \rangle$, where A is a set, C is a ternary relation on A , \smile is a unary operation on A , and $I \subseteq A$, is the Boolean algebra of all subsets of A supplemented with I as a distinguished element, the unary complex converse operation defined by $\check{X} = \{\check{x} : x \in X\}$ for all $X \subseteq A$, and the complex relative multiplication defined by

$$(3) \quad X; Y = \{z : \exists x \in X, \exists y \in Y, \langle x, y, z \rangle \in C\}$$

for all $X, Y \subseteq A$. Every complete atomic NA is isomorphic to the complex algebra of its atom structure [20, Th. 3.13(2)]. Therefore, to define a complete atomic NA, as we do in the following definition, it is enough to describe its atom structure.

Definition 5. Assume $\mathbf{E} \subseteq \mathbf{A} \in \mathbf{RA}$ are finite symmetric integral relation algebras. Then $C_{\mathbf{E}}(\mathbf{A})$ is the complete atomic NA with this atom structure: the atoms of $C_{\mathbf{E}}(\mathbf{A})$ are $1'$ and the ordered pair $x^{(i)}$ for every diversity atom x of \mathbf{A} and every index $i \in \omega$,

$$(4) \quad At(C_{\mathbf{E}}(\mathbf{A})) := \{1'\} \cup \{x^{(i)} : 0' \geq x \in At(\mathbf{A}), i \in \omega\},$$

the converse of every atom is itself, if $T \subseteq \omega^3$ is defined for $i, j, k \in \omega$ by

$$T(i, j, k) \iff (i \leq j = k) \vee (j \leq k = i) \vee (k \leq i = j),$$

and $c(x)$ is the atom of \mathbf{E} containing the atom x of \mathbf{A} , then the cycles of $C_{\mathbf{E}}(\mathbf{A})$ are, for all $0' \geq x, y, z \in At(\mathbf{A})$ and $i, j, k \in \omega$,

$$(5) \quad [1', 1', 1'], \quad [1', x^{(i)}, x^{(i)}],$$

$$(6) \quad [x^{(i)}, y^{(j)}, z^{(k)}] \quad \text{if } x; y \geq z \wedge (c(x) = c(y) = c(z) \Rightarrow T(i, j, k)).$$

For any $a \in A$ and $n \in \omega$, define the element $J(a, n)$ of $C_{\mathbf{E}}(\mathbf{A})$ by

$$(7) \quad J(a, n) = \sum \{x^{(i)} : 0' \cdot a \geq x \in At(\mathbf{A}), n \leq i \in \omega\} + \sum \{1' : 1' \leq a\},$$

so if x is an atom of \mathbf{A} , then

$$(8) \quad J(x, n) = \begin{cases} \sum \{x^{(i)} : n \leq i \in \omega\} & \text{if } x \leq 0' \\ 1' & \text{if } x = 1' \end{cases},$$

(i) if $0' \geq x \in At(\mathbf{A})$, $c(x) = a \in At(\mathbf{E})$, $i, j \in \omega$, and $i \neq j$, then

$$x^{(i)}; x^{(j)} = J(0' \cdot \bar{a} \cdot x; x, 0) + \sum \{z^{(k)} : k \leq i, a \cdot x; x \geq z \in At(\mathbf{A})\} + 1',$$

$$x^{(i)}; x^{(j)} = J(0' \cdot \bar{a} \cdot x; x, 0) + \sum \{z^{(\max(i, j))} : a \cdot x; x \geq z \in At(\mathbf{A})\},$$

(ii) if $0' \geq x, y \in At(\mathbf{A})$, $i, j \in \omega$, and $c(x) \neq c(y)$ then

$$x^{(i)}; y^{(j)} = J(x; y, 0) = \sum \{z^{(k)} : x; y \geq z \in At(\mathbf{A}), k \in \omega\},$$

(iii) if $0' \geq x, y \in At(\mathbf{A})$, $x \neq y$, $i, j \in \omega$, $i \neq j$, and $c(x) = c(y) = a \in At(\mathbf{E})$ then

$$x^{(i)}; y^{(j)} = J(0' \cdot \bar{a} \cdot x; y, 0) + \sum \{z^{(\max(i, j))} : a \cdot x; y \geq z \in At(\mathbf{A})\},$$

$$x^{(i)}; y^{(j)} = J(0' \cdot \bar{a} \cdot x; y, 0) + \sum \{z^{(k)} : k \leq i, a \cdot x; y \geq z \in At(\mathbf{A})\}.$$

Start with a finite symmetric integral relation algebra \mathbf{A} in which every atom is splittable (in the sense of [1]). Let $\mathbf{A}^\omega \supseteq \mathbf{A}$ be the relation algebra obtained by splitting every atom $a \in At(\mathbf{A})$ into ω pieces $a^{(0)}, a^{(1)}, \dots$ so that $a = \sum_{i \in \omega} a^{(i)}$. Splitting produces the maximum set of cycles in the extension $\mathbf{A}^\omega \supseteq \mathbf{A}$ that are consistent with containing \mathbf{A} as a subalgebra. Let $\mathbf{E} \subseteq \mathbf{A}$ be a subalgebra of \mathbf{A} . From the atom structure of \mathbf{A}^ω we obtain a new atom structure whose complex algebra is, in fact, isomorphic to $C_{\mathbf{E}}(\mathbf{A})$, by deleting all the diversity cycles $[a^{(i)}, b^{(j)}, c^{(k)}]$ of \mathbf{A}^ω which have the property that all the atoms in the cycle lie below the same atom of \mathbf{E} , and $T(i, j, k)$ fails to hold. This leaves only a “thin” remnant of the cycles of \mathbf{A}^ω that we would classify as “1-cycles of \mathbf{E} ” (because their atoms all lie below a single atom of \mathbf{E}). The set of 1-cycles produced by splitting is significantly reduced by imposing the “thinning condition” $T(i, j, k)$. Those cycles of \mathbf{A} that are “covered” by 1-cycles of \mathbf{E} are “thinly reproduced” in $C_{\mathbf{E}}(\mathbf{A})$, while the 2- and 3-cycles of \mathbf{A} that are covered by 2- or 3-cycles of \mathbf{E} are “split” into as many cycles as possible. Treating 1-, 2-, and 3-cycles differently in various combinations, either thinning or splitting each type of cycle, gives six more constructions that perhaps should be examined with regard to Problem 1.

Lemma 2. \mathbf{A} is isomorphic, by $a \mapsto J(a, n)$, to a subalgebra \mathbf{A}' of $C_{\mathbf{E}}(\mathbf{A})$.

$$\mathbf{A} \cong \mathbf{A}' \subseteq C_{\mathbf{E}}(\mathbf{A}).$$

Proof. Define the function $\varphi : \mathbf{A} \rightarrow C_{\mathbf{E}}(\mathbf{A})$ by $\varphi(a) = J(a, 0)$ for all $a \in A$. For a key part of the proof that φ embeds \mathbf{A} into $C_{\mathbf{E}}(\mathbf{A})$, assume $0' \geq x, y \in At(\mathbf{A})$. We wish to prove that $\varphi(x); \varphi(y)$ and $\varphi(x; y)$ contain the same diversity atoms of $C_{\mathbf{E}}(\mathbf{A})$. (Proofs for the other parts, involving preservation by φ of the Boolean structure and identity element, are fairly easy.)

Consider an arbitrary diversity atom $z^{(k)} \in At(C_{\mathbf{E}}(\mathbf{A}))$, where $0' \geq z \in At(\mathbf{A})$, $k \in \omega$. Assume $z^{(k)} \leq \varphi(x); \varphi(y)$. Then there are $u, v \in At(C_{\mathbf{E}}(\mathbf{A}))$ such that

$z^{(k)} \leq u;v$, $\varphi(x) \geq u \in \text{At}(C_{\mathbf{E}}(\mathbf{A}))$, and $\varphi(y) \geq v \in \text{At}(C_{\mathbf{E}}(\mathbf{A}))$. By (8) there are some $i, j \in \omega$ such that $u = x^{(i)}$ and $v = y^{(j)}$. But then $[x^{(i)}, y^{(j)}, z^{(k)}]$ is a cycle of $C_{\mathbf{E}}(\mathbf{A})$, so $x; y \geq z$ in \mathbf{A} , which implies $z^{(k)} \leq J(x; y, 0)$, hence $z^{(k)} \leq \varphi(x; y)$. The argument is reversible. \square

Every diversity atom in \mathbf{A}' is the join of an infinite set of atoms. Therefore $C_{\mathbf{E}}(\mathbf{A})$ cannot be a relation algebra if \mathbf{A} has any diversity atoms that are not splittable. In fact, $C_{\mathbf{E}}(\mathbf{A})$ satisfies all the axioms for relation algebras except possibly the associative law, so $C_{\mathbf{E}}(\mathbf{A}) \in \text{NA}$.

Here is a computational lemma needed several times later.

Lemma 3. *Assume \mathbf{A} is a special extension of \mathbf{E} , a, b are distinct diversity atoms of \mathbf{E} , and u, v are diversity atoms of $C_{\mathbf{E}}(\mathbf{A})$. If $u \leq J(a, 0)$ and $v \leq J(b, 0)$ then $u;v = J(a; b, 0)$. In particular, if $a; b = 0'$, then $u;v = 0'$.*

Proof. From $u \leq J(a, 0)$ and $v \leq J(b, 0)$ we have $x^{(i)} = u$, $y^{(j)} = v$, $x \leq a = c(x)$, and $y \leq b = c(y)$, for some $x, y \in \text{At}(\mathbf{A})$ and $i, j \in \omega$. The covers of x and y are different because $x \leq c(x) = a \neq b = c(y) \geq y$. Rule Def. 5(ii) applies in this case and says that $x^{(i)}; y^{(j)} = J(x; y, 0)$. Note that $x; y \leq a; b$. Since \mathbf{A} is a special extension of \mathbf{E} , we deduce from Def. 4(i) that every atom of \mathbf{E} below $a; b$ is also below $x; y$, hence $x; y = a; b$. We conclude that $u;v = x^{(i)}; y^{(j)} = J(x; y, 0) = J(a; b, 0)$. If $a; b = 0'$, then $u;v = J(0', 0)$, but $J(0', 0)$ is the diversity element $0'$ of $C_{\mathbf{E}}(\mathbf{A})$, so $u;v = 0'$. \square

4. Embedding Monk algebras

Two elements of an atomic relation algebra are said to be **almost the same** if their symmetric difference is the join of finitely many atoms. We show in Theorem 1 below that if \mathbf{A} is a special extension of \mathbf{E} and \mathbf{B} is the subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ generated by the atoms of $C_{\mathbf{E}}(\mathbf{A})$, then the finitely generated subalgebras of \mathbf{B} are finite and every element of \mathbf{B} almost the same as an element of the subalgebra \mathbf{E}' of \mathbf{B} isomorphic to \mathbf{E} by $a \mapsto J(a, 0)$.

For an example, suppose \mathbf{A} is a Monk algebra obtained from \mathbf{E}_q^{23} by splitting, $4 \leq q \in \omega$, and \mathbf{E} is a subalgebra of \mathbf{E}_q^{23} . By Lemma 1, \mathbf{A} is a special extension of \mathbf{E} , so Theorem 1 applies to \mathbf{A} and \mathbf{E} . Next, we show in Theorem 2(i)(ii) that if, in addition, \mathbf{E} has a “flexible trio” (Def .8 below) then \mathbf{B} is representable because every finitely generated subalgebra of \mathbf{B} is included in a finite subalgebra of \mathbf{B} that has the “1-point extension property” (Def .6 below). In the example, if $7 \leq q$ (\mathbf{A} has at least six colors) then \mathbf{E}_q^{23} has a subalgebra \mathbf{E} with a flexible trio, so Theorem 2(i)(ii) applies, and we conclude that $\mathbf{B} \in \text{RRA}$. Finally, we show in Theorem 2(iii) that if \mathbf{A} has no 1-cycles then \mathbf{B} is not completely representable and $C_{\mathbf{E}}(\mathbf{A})$, the completion of \mathbf{B} , is not representable. Theorem 2(iii) applies to \mathbf{A} because Monk algebras have no 1-cycles. Cor.1 accordingly says that every finite Monk algebra with six or more colors is a subalgebra of the non-representable completion of an atomic representable relation algebra whose finitely-generated subalgebras are finite.

The conclusion that $C_{\mathbf{E}}(\mathbf{A}) \notin \text{RRA}$ can be obtained without Theorem 2(iii) in case the Monk algebra \mathbf{A} is non-representable, which happens if the number of atoms is large compared to the number of colors. In this case the completion of \mathbf{B} is non-representable simply because it has a non-representable subalgebra (isomorphic to the non-representable Monk algebra \mathbf{A}).

In Theorem 1 we obtain conclusions just from knowing the extension $\mathbf{E} \subseteq \mathbf{A}$ is special. Then in Theorem 2 we also consider what happens when, in addition, \mathbf{E} has a flexible trio and \mathbf{A} has no 1-cycles.

Theorem 1. *Assume \mathbf{A} and \mathbf{E} are finite symmetric integral relation algebras, \mathbf{A} is a special extension of $\mathbf{E} \subseteq \mathbf{A}$, and $\mathbf{B} \subseteq C_{\mathbf{E}}(\mathbf{A})$ is the subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ generated by $At(C_{\mathbf{E}}(\mathbf{A}))$.*

- (i) \mathbf{B} is countable, atomic, symmetric, integral, and generated by its atoms.
- (ii) $C_{\mathbf{E}}(\mathbf{A})$ and \mathbf{B} have the same atom structure.
- (iii) $C_{\mathbf{E}}(\mathbf{A})$ is isomorphic to the complex algebra of the atom structure of \mathbf{B} .
- (iv) $C_{\mathbf{E}}(\mathbf{A})$ is the completion of \mathbf{B} .
- (v) There are subalgebras $\mathbf{E}' \subseteq \mathbf{A}' \subseteq C_{\mathbf{E}}(\mathbf{A})$ with $\mathbf{E}' \cong \mathbf{E}$ and $\mathbf{A}' \cong \mathbf{A}$.
- (vi) Every finitely generated subalgebra of \mathbf{B} is finite.
- (vii) Every element of \mathbf{B} is almost the same as an element of \mathbf{E}' .

Proof. Parts (i)–(iv) require only the assumption that $C_{\mathbf{E}}(\mathbf{A})$ is complete and atomic and \mathbf{B} is the subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ generated by the atoms of $C_{\mathbf{E}}(\mathbf{A})$. Everything in parts (i)–(iv) is either obvious or very easy to prove; see [20, Th. 3.13] for part (iii). Part (v) was proved in Lemma 2. The assumption that \mathbf{A} is a special extension of \mathbf{E} is needed only for Lemma 4 below, which is used to prove parts (vi) and (vii).

Lemma 4. *For every $n \in \omega$, B_n is the set of atoms of a subalgebra of $C_{\mathbf{E}}(\mathbf{A})$, where*

$$(9) \quad B_n = \{1'\} \cup \{x^{(i)} : 0' \geq x \in At(\mathbf{A}), n > i \in \omega\} \cup \{J(a, n) : 0' \geq a \in At(\mathbf{E})\}.$$

Proof. The elements of B_n are disjoint and their join is 1, so the set of joins of subsets of B_n is closed under the Boolean operations of $C_{\mathbf{E}}(\mathbf{A})$ and, under those operations, forms a Boolean algebra whose set of atoms is B_n . The converse of everything in B_n is again in B_n because conversion is the identity function on $C_{\mathbf{E}}(\mathbf{A})$. What remains is to show the relative product $u;v$ of any two elements $u, v \in B_n$ is the join of a subset of B_n . For this it is enough to show that every element $w \in B_n$ is contained in or disjoint from $u;v$. This is clearly true whenever $u = 1'$ or $v = 1'$ or w is itself an atom of $C_{\mathbf{E}}(\mathbf{A})$, so we may assume $w = J(a, n)$, for some $a \in At(\mathbf{E})$, and $u + v \leq 0'$. We will show that if $u;v$ has nonempty intersection with $J(a, n)$ then $u;v$ contains $J(a, n)$.

Suppose $u;v \cdot J(a, n) \neq 0$, where $0' \geq u, v \in B_n$, $0' \geq a \in At(\mathbf{E})$. Then there are $x, y, z \in At(\mathbf{A})$ and $i, j, k \in \omega$ such that $x^{(i)} \leq u$, $y^{(j)} \leq v$, $z^{(k)} \leq J(a, n)$, and $x^{(i)};y^{(j)} \geq z^{(k)}$. For both cases below, note that $c(z) = a$ and $n \leq k$ by Def. (7), and $x;y \geq z$ by (6).

Case 1. $\text{not } (c(x) = c(y) = c(z))$. From $x;y \geq z$ we get $x;y \cdot c(z) \neq 0$ since $0 \neq z \leq c(z)$, hence $x;y \geq c(z) = a$ by Def. 4(i). The implication in (6) has a false hypothesis and therefore holds trivially in this case for every atom of \mathbf{A} below a . It follows by (6) and Def. (7) that $x^{(i)};y^{(j)} \geq J(a, 0) \geq J(a, n)$.

Case 2. $c(x) = c(y) = c(z) = a$. In this case, by $x^{(i)};y^{(j)} \geq z^{(k)}$ and (6) we have $T(i, j, k)$. From $x^{(i)} \leq u \in B_n$ and the relevant definitions, it follows that if u is an atom of $C_{\mathbf{E}}(\mathbf{A})$ then $u = x^{(i)}$ and $i < n$, while if u is not an atom of $C_{\mathbf{E}}(\mathbf{A})$, then $x^{(i)} \leq u = J(a, n)$ since $a = c(x)$, and $i \geq n$. Consequently, if both u and v were atoms of $C_{\mathbf{E}}(\mathbf{A})$, we would have $i, j < n \leq k$, contrary to $T(i, j, k)$. Hence either $u = J(a, n)$ or $v = J(a, n)$. Since $C_{\mathbf{E}}(\mathbf{A})$ is symmetric, these are

really the same case. We assume $u = x^{(i)}$ and $v = J(a, n)$, and will prove that $x^{(i)}; J(a, n) \geq J(a, n)$.

Toward this end, assume $w_j \leq J(a, n)$ where $n \leq j \in \omega$ and $w \leq a = c(w)$. Now $x^{(i)}$ is in B_n , so $i < n \leq j$, hence $T(i, j, j)$. From $x; y \geq z$ and $c(x) = c(y) = c(z) = a$ we get $a; a \cdot a \neq 0$, but $x \leq a$ and $w \leq a$, so $x; w \cdot a \neq 0$ by Def. 4(ii). We may therefore choose an atom $t \in \mathbf{A}$ such that $t \leq x; w \cdot a$. Then $c(t) = a$ and $T(i, j, j)$, so $[x^{(i)}, t^{(j)}, w^{(j)}]$ is a cycle of $C_{\mathbf{E}}(\mathbf{A})$ by (6), and $t^{(j)} \leq J(a, n)$ since $t \leq a$ and $n \leq j$, so $w_j \leq x^{(i)}; t^{(j)} \leq x^{(i)}; J(a, n)$. Since this holds for all atoms w_j below $J(a, n)$, we have proved $J(a, n) \leq x^{(i)}; J(a, n)$.

We have shown that every product of two elements of B_n is the join of a subset of B_n . It follows that $u; v = \sum \{w : u; v \geq w \in B_n\}$ for all $u, v \in B_n$. Hence, for all $U, V \subseteq B_n$, we have

$$\begin{aligned} \sum U; \sum V &= \sum \{u; v : u \in U, v \in V\} \\ &= \sum \{\sum \{w : u; v \geq w \in B_n\} : u \in U, v \in V\} \\ &= \sum \{w : u; v \geq w \in B_n, u \in U, v \in V\} \\ &\in \{\sum X : X \subseteq B_n\}. \end{aligned}$$

Therefore $\{\sum X : X \subseteq B_n\}$ is closed under relative multiplication and is a subalgebra of $C_{\mathbf{E}}(\mathbf{A})$. \square

We return to the proof of Theorem 1. Suppose \mathbf{F} is a finitely generated subalgebra of \mathbf{B} . Since \mathbf{B} is itself generated by $At(C_{\mathbf{E}}(\mathbf{A}))$, there is a finite set of atoms $X \subseteq At(C_{\mathbf{E}}(\mathbf{A}))$ such that \mathbf{F} is contained in the subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ generated by X . Since X is finite and $At(C_{\mathbf{E}}(\mathbf{A})) \subseteq \bigcup_{n \in \omega} B_n$, we may choose a sufficiently large $n \in \omega$ so that $X \subseteq B_n$. Then \mathbf{F} is contained in the subalgebra \mathbf{B}_n of $C_{\mathbf{E}}(\mathbf{A})$ generated by B_n . The subalgebra \mathbf{B}_n is finite by the lemma, since its set of atoms is the finite set B_n , so \mathbf{F} is also finite. Hence (vi) holds. This argument also shows that every element of \mathbf{B} is, for some $n \in \omega$, included in a subalgebra whose set of atoms is B_n , and hence is a join of elements of B_n . But every join of elements of B_n is almost the same as one of the atoms of \mathbf{E}' . Hence (vii) holds. \square

As it happens, every finitely-generated subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ (not just \mathbf{B}) is also finite, even if the extension $\mathbf{E} \subseteq \mathbf{A}$ is not special. To prove this, one argues that for every finite subset F of $C_{\mathbf{E}}(\mathbf{A})$ there is some $n \in \omega$ and some finite partition \mathcal{P} of $\{i : n \leq i \in \omega\}$ such that

$$\{1'\} \cup \{x^{(i)} : x \in At(\mathbf{A}), n > i \in \omega\} \cup \{\sum \{x^{(i)} : i \in P\} : x \in At(\mathbf{A}), P \in \mathcal{P}\}$$

is the set of atoms of a subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ that contains F . The remaining details of this proof are omitted since this fact is not needed and it is also not in itself enough to prove Lemma 4. On the other hand, there is a special case which is easy to prove and needed later.

Lemma 5. *Assume $\mathbf{A} \supseteq \mathbf{E}$ are finite symmetric integral relation algebras. For every $n \in \omega$,*

$$(10) \quad \{1'\} \cup \{x^{(i)} : 0' \geq x \in At(\mathbf{A}), n > i \in \omega\} \cup \{J(a, n) : 0' \geq a \in At(\mathbf{A})\}.$$

is the set of atoms of a subalgebra of $C_{\mathbf{E}}(\mathbf{A})$.

Proof. The proof is similar to, but simpler, than the proof of Lemma 4. The closure of the set of joins of subsets of (10) under relative multiplication is an immediate consequence of Def. 5(i)(ii)(iii). \square

For the next theorem we need some definitions. A relation algebra has the 1-point extension property if, loosing speakly, every “finite partial representation” μ can be extended by one point wherever this is needed. We make this precise as follows.

Definition 6. For any $k \in \omega$ and any atomic relation algebra \mathbf{A} , $B_k(\mathbf{A})$ is the set of functions $\mu : k \times k \rightarrow \text{At}(\mathbf{A})$ that satisfy the following conditions.

- (B₀) $\mu_{i,i} \leq 1'$ for all $i < k$,
- (B₁) $\check{\mu}_{i,j} = \mu_{j,i}$ for all $i, j < k$,
- (B₂) $\mu_{i,l}; \mu_{l,j} \geq \mu_{i,j}$ for all $i, j, l < k$.

The elements of $B_k(\mathbf{A})$ are called **basic matrices**. A matrix μ satisfies the **identity condition** if $\mu_{l,m} = 1'$ iff $l = m$ for all $l, m < k$. We say that \mathbf{A} has the **1-point extension property** if, assuming $\mu \in B_k(\mathbf{A})$, μ satisfies the identity condition, x, y are diversity atoms of \mathbf{A} , $i, j < k$, $i \neq j$, and $\mu_{i,j} \leq x; y$, there are basic matrix $\mu' \in B_{k+1}(\mathbf{A})$ satisfying the identity condition such that $\mu_{l,m} = \mu'_{l,m}$ for all $l, m < k$, $\mu'_{i,k} = x$, and $\mu'_{k,j} = y$.

Definition 7. A relation algebra \mathbf{A} is **completely representable** if it has a complete representation, where a representation ρ , mapping \mathbf{A} into some algebra of binary relations, is **complete** if it preserves all joins, i.e., if X is a subset of \mathbf{A} whose join $\sum X$ exists in \mathbf{A} , then $\rho(\sum X) = \bigcup_{x \in X} \rho(x)$.

Definition 8. Three diversity atoms a, b, c of a symmetric integral relation algebra \mathbf{A} are said to be a **flexible trio** if

$$(11) \quad a; a = b; b = c; c = 1,$$

$$(12) \quad a; b = a; c = b; c = 0',$$

and, for every atom $x \notin \{1', a, b, c\}$,

$$(13) \quad x; a = x; b = 0' \vee x; a = x; c = 0' \vee x; b = x; c = 0'.$$

Theorem 4 (relegated to an Appendix) shows that having a flexible trio is sufficient for representability.

If each of a , b , and c is a flexible atom then a, b, c is a flexible trio. Hence any three diversity atoms of \mathbf{E}_q^{23} form a flexible trio. It can happen that a, b, c is a flexible trio but none of a, b, c is flexible, as in, for example, the symmetric integral relation algebra with seven atoms $1', a, b, c, d, e, f$ and all diversity cycles *except* $[a, d, d]$, $[b, e, e]$, and $[c, f, f]$. This algebra has no flexible atoms.

The following theorem applies to all Monk algebras with at least six colors, but many other algebras also satisfy its hypotheses. For an example, let \mathbf{A} be the symmetric integral relation algebra whose atoms are $1', a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$, and c_3 , and whose diversity cycles consist of none of the 1-cycles, all of the 2-cycles, and all the 3-cycles *except* $[a_1, a_2, a_3]$, $[b_1, b_2, b_3]$, and $[c_1, c_2, c_3]$. Let \mathbf{E} be the subalgebra of \mathbf{A} whose atoms are $1', a = a_1 + a_2 + a_3, b = b_1 + b_2 + b_3$, and $c = c_1 + c_2 + c_3$. Then \mathbf{A} is a special extension of \mathbf{E} , a, b, c is a flexible trio of \mathbf{E} , and \mathbf{A} has no 1-cycles, so Th. 2 can applied to conclude that \mathbf{A} is in the class K of Problem 1, but \mathbf{A} is not a Monk algebra. More than 3000 additional examples can

be obtained by deleting any or all of the following 2-cycles: $[a_1, a_2, a_2]$, $[a_2, a_3, a_3]$, $[a_1, a_1, a_3]$, $[b_1, b_2, b_2]$, $[b_2, b_3, b_3]$, $[b_1, b_1, b_3]$, $[c_1, c_2, c_2]$, $[c_2, c_3, c_3]$, $[c_1, c_1, c_3]$, and restoring any or all of the deleted 3-cycles, but at least one of the 2- or 3-cycles must be deleted to insure \mathbf{A} is not a Monk algebra. This scheme retains enough 2-cycles that the extension is always special.

Theorem 2. *Assume $\mathbf{E} \subseteq \mathbf{A}$ are finite symmetric integral relation algebras, \mathbf{A} is a special extension of \mathbf{E} , and \mathbf{E} has a flexible trio. If \mathbf{B} is the subalgebra of $C_{\mathbf{E}}(\mathbf{A})$ generated by $At(C_{\mathbf{E}}(\mathbf{A}))$, then*

- (i) *every finitely generated subalgebra of \mathbf{B} is contained in a subalgebra of \mathbf{B} that has the 1-point extension property,*
- (ii) *\mathbf{B} is representable,*
- (iii) *if \mathbf{A} has no 1-cycles, i.e., $u;u \cdot u = 0$ whenever $1' \neq u \in At(\mathbf{A})$, then \mathbf{B} is not completely representable and the completion of \mathbf{B} is not representable.*

Proof. Suppose \mathbf{F} is a finitely generated subalgebra of \mathbf{B} . By the argument at the end of the proof of Th. 1, there is some $n \in \omega$ such that \mathbf{F} is contained in the subalgebra $\mathbf{B}_n \subseteq \mathbf{B}$ with atoms $At(\mathbf{B}_n) = B_n$. Let a, b, c be a flexible trio of \mathbf{E} . We will show that $J(a, n), J(b, n), J(c, n)$ is a flexible trio of \mathbf{B}_n . Consider the product of the first two elements of the trio. Note that $J(a, n); J(b, n) \leq 0'$ since $J(a, n)$ and $J(b, n)$ are disjoint atoms of \mathbf{B}_n . We have

$$J(a, n); J(b, n) = \sum \{u;v : J(a, n) \geq u \in At(C_{\mathbf{E}}(\mathbf{A})), J(b, n) \geq v \in At(C_{\mathbf{E}}(\mathbf{A}))\}$$

but every disjunct $u;v$ in this last join is $0'$ by Lemma 3 and the assumption $a;b = 0'$, so $J(a, n); J(b, n) = 0'$. Similarly, $J(a, n); J(c, n) = 0' = J(b, n); J(c, n)$. Thus (12) holds.

For (13), consider a diversity atom of \mathbf{B}_n that is not one of $J(a, n), J(b, n), J(c, n)$. It is either an atom of $C_{\mathbf{E}}(\mathbf{A})$ or an atom of \mathbf{B}_n with the form $J(d, n)$, where d is a diversity atom of \mathbf{E} distinct from a, b, c .

We first consider $J(d, n)$. Now d multiplies to $0'$ with two of a, b, c by (13), say $a;d = b;d = 0'$. Choose atoms x, y of \mathbf{A} with $x \leq a$ and $y \leq d$. Then $J(a, n); J(d, n) \leq 0'$ since $J(a, n)$ and $J(d, n)$ are disjoint, and $J(a, n); J(d, n) \geq x^{(n)}; y^{(n)}$, but $x^{(n)}; y^{(n)} = 0'$ by Lemma 3 because $a;d = 0'$, so $J(a, n); J(d, n) = 0'$. Similarly $J(b, n); J(d, n) \geq 0'$, so the atom $J(d, n)$ multiplies to $0'$ with two of $J(a, n), J(b, n), J(c, n)$, as desired.

Next consider an atom u of $C_{\mathbf{E}}(\mathbf{A})$. It has the form $u = x^{(i)}$ for some diversity atom x of \mathbf{A} and some $i < n$. We claim that the product of $c(x)$ with (at least) two elements in the trio a, b, c is $0'$, say $a;c(x) = b;c(x) = 0'$. This follows from (13) if $c(x)$ is a diversity atom distinct from a, b, c , but if $c(x)$ is one of a, b, c , then it follows from (12). Choose an atom $a \geq y \in At(\mathbf{A})$. Then $x^{(i)}; J(a, n) \geq x^{(i)}; y^{(n)} = 0'$ by $a;d = 0'$ and Lemma 3. Similarly $x^{(i)}; J(b, n) = 0'$, so the atom $u = x^{(i)}$ multiplies to $0'$ with two of $J(a, n), J(b, n), J(c, n)$, as desired. This finishes the proof of (13) for $J(a, n), J(b, n), J(c, n)$.

For (11), we will prove $J(a, n); J(a, n) = 1$ from $a;a = 1$. Assume $u = x^{(i)} \in At(C_{\mathbf{E}}(\mathbf{A}))$, $0' \geq x \in At(\mathbf{A})$, and $0 \leq i \in \omega$. Then $x \leq 1 = a;a = \sum \{y;z : a \geq y, z \in At(\mathbf{A})\}$ so there are atoms $a \geq y, z \in At(\mathbf{A})$ such that $x \leq y; z$. Note that $c(y) = c(z) = a$. Choose any j such that $\max(i, n) \leq j \in \omega$. Then $T(i, j, j)$ holds, so $[x^{(i)}, y^{(j)}, z^{(j)}]$ is a cycle of $C_{\mathbf{E}}(\mathbf{A})$ by (6), hence $u = x^{(i)} \leq y^{(j)}; z^{(j)} \leq$

$J(a, n); J(b, n)$. This shows $J(a, n); J(a, n) = 1$, and we obtain $J(b, n); J(b, n) = 1 = J(c, n); J(c, n)$ similarly from $b; b = c; c = 1$.

This completes the proof that $J(a, n), J(b, n), J(c, n)$ is a flexible trio of \mathbf{B}_n . By Theorem 4 below, \mathbf{B}_n has the 1-point extension property and is therefore representable. Every finitely generated subalgebra of \mathbf{B} is representable, hence \mathbf{B} is representable since RRA is a variety. Thus parts (i) and (ii) hold.

For part (iii), assume such that $u; u \cdot u = 0$ whenever $1' \neq u \in At(\mathbf{A})$. Suppose that ρ is a complete representation of \mathbf{B} . By definition, a complete representation preserves all joins. In particular, the join of the diversity atoms is $0'$, so the union of the representations of the diversity atoms must be the diversity relation, consisting of all pairs of distinct objects. Thus every pair $\langle i, j \rangle$ with $i \neq j$ is in the representation of a diversity atom of \mathbf{B} . But $\mathbf{A} \cong \mathbf{A}' \subseteq \mathbf{B}$ via $a \mapsto J(a, 0)$, so the representation ρ determines a coloring of the edges of K_ω (the complete graph on countably many vertices) as follows: the “color” of the edge (i, j) is $a \in At(\mathbf{A})$ if $\langle i, j \rangle \in \rho(J(a, 0))$. This coloring has no monochrome triangles because \mathbf{A} has no 1-cycles, contrary to Berge [3, Prop. 1, p. 440]. Therefore there is no such ρ and \mathbf{B} is not completely representable. \square

Corollary 1. *If \mathbf{A} is a finite Monk algebra with six or more colors then \mathbf{A} is in the class K defined in Problem 1. In fact, \mathbf{A} is a subalgebra of the completion of a relation algebra \mathbf{B} such that*

- \mathbf{B} is a countable, atomic, symmetric, integral relation algebra that is generated by its atoms,
- every finitely generated subalgebra of \mathbf{B} is contained in a finite subalgebra of \mathbf{B} with the 1-point extension property,
- the completion of \mathbf{B} has the same atom structure as \mathbf{B} , is isomorphic to the complex algebra of the atom structure of \mathbf{B} , and is not representable,
- \mathbf{B} is representable but not completely representable.

The smallest example to which these considerations apply is $\mathbf{A} = \mathbf{E}_7^{23}$, the example considered earlier. This algebra is a Monk algebra with six colors and no 1-cycles, obtained from itself by splitting. By the Street-Whitehead-Comer method, $\mathbf{E}_7^{23} \in \text{RRA}$ because \mathbf{E}_7^{23} has square representations on sets containing 97, 157, and 277 elements. [AMS Meeting, Iowa City, March 2011.] Suppose the diversity atoms of \mathbf{E}_7^{23} are $\mathbf{e}_1, \dots, \mathbf{e}_6$. Let $\mathbf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{a}_2 = \mathbf{e}_3 + \mathbf{e}_4$, and $\mathbf{a}_3 = \mathbf{e}_5 + \mathbf{e}_6$. Then \mathbf{A} is special extension of the subalgebra \mathbf{E} whose atoms are $1', \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is a flexible trio of individually flexible atoms in \mathbf{E} . Then RRA is not closed under completions because $C_{\mathbf{E}}(\mathbf{E}_7^{23})$ is the non-representable completion of the atomic representable subalgebra of $C_{\mathbf{E}}(\mathbf{E}_7^{23})$ generated by $At(C_{\mathbf{E}}(\mathbf{E}_7^{23}))$. In the next section we compute the exact degree of non-representability of $C_{\mathbf{E}}(\mathbf{E}_7^{23})$.

5. Cylindric algebras

CA_n is the class of n -dimensional cylindric algebras. Given a cylindric algebra $\mathbf{D} \in \text{CA}_n$ of dimension $n \geq 3$, the **relation algebraic reduct** $\text{Ra}(\mathbf{D})$ is defined in [8, Def. 5.3.7] and is a relation algebra if $n \geq 4$ by [8, Def. 5.3.8]. For any class $K \subseteq \text{CA}_n$ with $3 \leq n$, let $\text{Ra}K$ be the class of relation-algebraic reducts of subalgebras of neat 3-dimensional reducts of algebras in K :

$$\text{Ra}K = \text{Ra}^* \text{SNr}_3 K$$

By [8, 5.3.9, 5.3.16, 5.3.17], we have

$$\text{RRA} = \prod_{n \in \omega} \text{RaCA}_{n+4} \subseteq \cdots \subseteq \text{RaCA}_5 \subseteq \text{RaCA}_4 = \text{RA}.$$

Every non-representable relation algebra lies somewhere on this chain. The location of the example $C_{\mathbf{E}}(\mathbf{E}_7^{23})$ is determined by the main result in this section, which implies

$$(14) \quad C_{\mathbf{E}}(\mathbf{E}_7^{23}) \in \text{RaCA}_7 \sim \text{RaCA}_8.$$

Definition 9. Assume $\mathbf{A} \in \text{NA}$ is atomic and $k \leq \omega$. Two basic matrices μ and μ' in $B_k(\mathbf{A})$ **agree up to i** if $\mu_{l,m} = \mu'_{l,m}$ whenever $i \neq l, m \in k$, and they **agree up to i, j** if $\mu_{l,m} = \mu'_{l,m}$ whenever $i, j \neq l, m \in k$. We say that $\mathcal{M} \subseteq B_k(\mathbf{A})$ is an **k -dimensional relational basis for \mathbf{A}** if

- (R₀) for every atom $a \in \text{At}(\mathbf{A})$ there is a basic matrix $\mu \in \mathcal{M}$ such that $\mu_{0,1} = a$,
- (R₁) if $\mu \in \mathcal{M}$, $i, j < k$, $x, y \in \text{At}\mathbf{A}$, $\mu_{i,j} \leq x; y$, and $i, j \neq l < k$, then there is some $\mu' \in \mathcal{M}$ such that μ and μ' agree up to l , $\mu'_{i,l} = x$, and $\mu'_{l,j} = y$.

For any $i, j < k$ let

$$T_i^k(\mathbf{A}) = \{ \langle \mu, \mu' \rangle \in B_k(\mathbf{A}) \times B_k(\mathbf{A}) : \mu \text{ and } \mu' \text{ agree up to } i \},$$

$$E_{i,j}^k(\mathbf{A}) = \{ \mu \in B_k(\mathbf{A}) : \mu_{i,j} \leq 1' \}.$$

We say that $\mathcal{M} \subseteq B_k\mathbf{A}$ is a **k -dimensional cylindric basis for \mathbf{A}** if

- (C₀) if $a, b, c \in \text{At}(\mathbf{A})$, and $a \leq b; c$, then there is a basic matrix $\mu \in \mathcal{M}$ such that $\mu_{01} = a$, $\mu_{02} = b$, and $\mu_{21} = c$,
- (C₁) if $\mu, \mu' \in \mathcal{M}$, $i, j < k$, $i \neq j$, and μ agrees with μ' up to i, j , then there is some $\mu'' \in \mathcal{M}$ such that μ'' agrees with μ up to i , and μ'' agrees with μ' up to j , i.e., $\langle \mu'', \mu \rangle \in T_i^k(\mathbf{A})$ and $\langle \mu'', \mu' \rangle \in T_j^k(\mathbf{A})$,
- (C₂) if $\mu \in \mathcal{M}$ and $i, j < k$ then $\mu[i/j] \in \mathcal{M}$, where $[i/j](m) = m$ if $i \neq m < k$, and $[i/j](i) = j$.

For every $\mathcal{M} \subseteq B_k(\mathbf{A})$, let

$$(15) \quad \text{Ca}(\mathcal{M}) = \text{Cm}(\langle \mathcal{M}, T_i, E_{ij} \rangle_{i,j < k})$$

be the complex algebra of the relational structure $\langle \mathcal{M}, T_i, E_{ij} \rangle_{i,j < k}$, as defined in [7, 2.7.33], where $E_{ij} = E_{ij}^k(\mathbf{A}) \cap \mathcal{M}$ and $T_i = T_i^k(\mathbf{A}) \cap (\mathcal{M} \times \mathcal{M})$ for all $i, j < k$.

Theorem 3. Assume $4 \leq r \in \omega$, $\mathbf{A} = \mathbf{E}_{r+3}^{23}$, and the atoms of \mathbf{A} are

$$\mathbf{e}_0 = 1', \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \dots, \mathbf{e}_{r+2}.$$

Then \mathbf{A} is a special extension of a subalgebra \mathbf{E} whose r atoms are

$$\begin{array}{llll} \mathbf{a}_0 = \mathbf{e}_0 = 1' & \mathbf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2 & \mathbf{a}_2 = \mathbf{e}_3 + \mathbf{e}_4 & \mathbf{a}_3 = \mathbf{e}_5 + \mathbf{e}_6 \\ & \mathbf{a}_4 = \mathbf{e}_7 & \dots & \mathbf{a}_{r-1} = \mathbf{e}_{r+2} \end{array}$$

and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is a flexible trio of \mathbf{E} , so by Ths.1,2, the atom-generated subalgebra of the complete atomic relation algebra $C_{\mathbf{E}}(\mathbf{A})$ is an atomic atom-generated symmetric integral representable relation algebra with finite finitely-generated subalgebras, and if $3 \leq n \leq r+3$ then

- (i) $B_n(C_{\mathbf{E}}(\mathbf{A}))$ is an n -dimensional cylindric basis for $C_{\mathbf{E}}(\mathbf{A})$,
- (ii) $\text{Ca}(B_n(C_{\mathbf{E}}(\mathbf{A})))$ is a complete atomic n -dimensional cylindric algebra,

- (iii) $C_{\mathbf{E}}(\mathbf{A})$ is isomorphic to the relation algebraic reduct of $\mathbf{Ca}(B_n(C_{\mathbf{E}}(\mathbf{A})))$ and $C_{\mathbf{E}}(\mathbf{A}) \in \text{RaCA}_n$,
- (iv) $\mathbf{Ca}(B_n(C_{\mathbf{E}}(\mathbf{A}))) \notin \text{SNr}_n \text{CA}_{r+4}$.

Proof. By [23, Th. 7], in order to prove $B_n(C_{\mathbf{E}}(\mathbf{A}))$ is a cylindric basis for $C_{\mathbf{E}}(\mathbf{A})$ it is enough to show, given $n-2$ pairs of diversity atoms $u_1, v_1, \dots, u_{n-2}, v_{n-2}$ of $C_{\mathbf{E}}(\mathbf{A})$, that $\prod_{1 \leq i \leq n-2} u_i; v_i \neq 0$. We will find a diversity atom w , such that w is included in every product $u_i; v_i$, $1 \leq i \leq n-2$. Any product $u_i; v_i$ that is equal to $0'$ or 1 imposes no restriction on our choice of w . We therefore assume that *none* of the products is $0'$ or 1 , i.e., $0' \neq u_i; v_i \neq 1$ whenever $1 \leq i \leq n-2$.

Consequently, for every product $u_i; v_i$ we know that there cannot be *distinct* atoms $a, b \in \text{At}(\mathbf{E})$ such that $u_i \leq J(a, 0)$ and $v_i \leq J(b, 0)$, because we would obtain $u_i; v_i = J(a; b, 0)$ from $a \neq b$ by Lemma 3, and a computation in \mathbf{E} shows $a; b = 0'$ since $a \neq b$, forcing $u_i; v_i = 0'$ in $C_{\mathbf{E}}(\mathbf{A})$, contrary to our assumption that no product is $0'$ or 1 . Therefore, there is a function $f : \{1, \dots, n-2\} \rightarrow \{1, \dots, r-1\}$ such that

$$(16) \quad u_i + v_i \leq J(\mathbf{a}_{f(i)}, 0) \text{ for all } i \in \{1, \dots, n-2\}.$$

We see next that every index in $\{1, \dots, r-1\}$ is in the range of f . Suppose some index $j \in \{1, \dots, r-1\}$ is not in the range of f . Consider any product $u_i; v_i$ with $1 \leq i \leq n-2$. Let $k = f(i)$ and note that $k \neq j$. There are atoms $x, y \in \text{At}(\mathbf{A})$ such that

$$(17) \quad x + y \leq \mathbf{a}_k = \mathbf{c}(x) = \mathbf{c}(y), \quad u_i \leq J(x, 0), \quad v_i \leq J(y, 0).$$

Since $k \neq j$, $\overline{\mathbf{a}}_k \geq \mathbf{a}_j$. By Def. 5(i)(iii) and (17), $u_i; v_i \geq J(0' \cdot \overline{\mathbf{a}}_k \cdot x; y, 0)$. If $x \neq y$ then $x; y = 0'$ in \mathbf{A} , so

$$(18) \quad u_i; v_i \geq J(0' \cdot \overline{\mathbf{a}}_k \cdot x; y, 0) = J(0' \cdot \overline{\mathbf{a}}_k, 0) \geq J(\mathbf{a}_j, 0).$$

If $x = y$ then $x; y = \overline{x} \geq \overline{\mathbf{a}}_k$ in \mathbf{A} , so $\overline{\mathbf{a}}_k \cdot x; y = \overline{\mathbf{a}}_k$, and again we have (18).⁴ Since (18) holds for every i , we obtain much more than $\prod_{i=1}^{n-2} u_i; v_i \neq 0$, in fact,

$$(19) \quad 0 \neq J(\mathbf{a}_j, 0) \leq \prod_{i=1}^{n-2} u_i; v_i.$$

Therefore, assume that f is surjective. Next we show that f is actually maps two distinct indices onto each of the atoms $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 , i.e., those atoms of \mathbf{E} that are the join of two atoms of \mathbf{A} . We prove this only for \mathbf{a}_1 . Since 1 is in the range of f , we'll suppose, for specificity and simplicity of notation, that $1 = f(1)$, i.e., $u_1 + v_1 \leq J(\mathbf{a}_1, 0)$. We wish to show that $1 = f(i)$ for some $i \neq 1$, so we assume this does *not* happen, i.e., assume $1 \neq f(i)$ for all $i \in \{2, \dots, n-2\}$. Now $\mathbf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2$, so there are atoms $x, y \in \{\mathbf{e}_1, \mathbf{e}_2\}$ and indices $k, l \in \omega$ such that $x + y \leq \mathbf{a}_1$, $u_1 = x^{(k)}$, and $v_1 = y^{(l)}$. Let $m = \max(k, l)$. Notice that $T(k, l, m)$ holds and \mathbf{E}_{r+3}^{23} contains the 2-cycles $[\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2]$ and $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2]$. It follows by (6) that

$$\begin{aligned} \mathbf{e}_1^{(k)}; \mathbf{e}_1^{(l)} &\geq \mathbf{e}_2^{(m)} \\ \mathbf{e}_1^{(k)}; \mathbf{e}_2^{(l)} &= \mathbf{e}_2^{(k)}; \mathbf{e}_1^{(l)} \geq \mathbf{e}_1^{(m)} + \mathbf{e}_2^{(m)} \\ \mathbf{e}_2^{(k)}; \mathbf{e}_2^{(l)} &\geq \mathbf{e}_1^{(m)} \end{aligned}$$

⁴By the way, we've shown $j \neq f(i) \implies u_i; v_i \geq J(\mathbf{a}_j, 0)$.

We may therefore let $w = \mathbf{e}_1^{(m)}$ if $x = y = \mathbf{e}_1$, $w = \mathbf{e}_2^{(m)}$ if $x = y = \mathbf{e}_2$, and either $w = \mathbf{e}_1$ or $w = \mathbf{e}_2$ if $x \neq y$. In every case, $u_1; v_1 \geq w$. For products other than $u_1; v_1$, note that if $2 \leq i \leq n-2$, then by our assumption we have $1 \neq f(i)$, hence by the footnote, $u_i; v_i \geq J(\mathbf{a}_1, 0) \geq w$. This shows $w \leq \prod_{i=2}^{n-2} u_i; v_i$, which, together with $w \leq u_1; v_1$, gives us $w \leq \prod_{i=1}^{n-2} u_i; v_i$.

At this point we know that either we are done because we have proved $\prod_{i=1}^{n-2} u_i; v_i \neq 0$, or else f maps at least one index from $\{1, \dots, n-2\}$ onto each of the indices in $\{4, \dots, r-1\}$, and f maps at least two indices from $\{1, \dots, n-2\}$ onto each of the indices in $\{1, 2, 3\}$. But $|\{1, \dots, n-2\}| = n-2$, $|\{4, \dots, r-1\}| = r-4$, and $|\{1, 2, 3\}| = 3$, so we must have $n-2 \geq r-4 + 2 \cdot 3 = r+2$, but our restriction on r is $n \leq r+3$, so $n-2 \leq r+1$, a contradiction. Therefore we do in fact know that $0 \neq \prod_{i=1}^{n-2} u_i; v_i$, as desired. This shows $B_n(C_{\mathbf{E}}(\mathbf{A}))$ is a cylindric basis for $C_{\mathbf{E}}(\mathbf{A})$ and completes the proof of part (i). Parts (ii) and (iii) follow from part (i) by [23, Theorem 10].

For part (iv), assume to the contrary that $\mathbf{Ca}(B_n(C_{\mathbf{E}}(\mathbf{A}))) \subseteq \mathbf{Nr}_n \mathbf{D}$ for some $\mathbf{D} \in \mathbf{CA}_{r+4}$. We get a contradiction by finding a subalgebra \mathbf{F} of $\mathbf{Ca}(B_n(C_{\mathbf{E}}(\mathbf{A})))$ which is not in $\mathbf{SNr}_n \mathbf{CA}_{r+4}$. From Theorem 5 with $p = r+3$ we get

$$(20) \quad \mathbf{Ca}(B_3(\mathbf{F})) \notin \mathbf{SNr}_3 \mathbf{CA}_{p+1} = \mathbf{SNr}_3 \mathbf{CA}_{r+4}$$

For this we choose an arbitrary finite parameter $N \in \omega$ and make it big enough. For this fixed N there is a finite subalgebra \mathbf{F} of $C_{\mathbf{E}}(\mathbf{A})$ whose atoms are $1'$, $\mathbf{e}_i^{(j)}$ and $J(\mathbf{e}_i, N)$ for $1 \leq i \leq r+2$ and $j < N$. This finite subalgebra \mathbf{F} has a subalgebra isomorphic to \mathbf{E}_{r+3}^{23} , whose atoms are $1'$ and $J(\mathbf{e}_i, 0)$ for $1 \leq i \leq r+2$. By Theorem 5 in the next section, a finite extension of \mathbf{E}_{r+3}^{23} with enough atoms satisfies (20). By choosing N large enough, the extension \mathbf{F} of \mathbf{E}_{r+3}^{23} has enough atoms. \square

REFERENCES

- [1] H. Andr  ka, R. D. Maddux, and I. N  meti, *Splitting in relation algebras*, Proc. Amer. Math. Soc. **111** (1991), no. 4, 1085–1093. MR 1052567 (91g:03126)
- [2] Hajnal Andr  ka, Istv  n N  meti, and Tarek Sayed Ahmed, *Omitting types for finite variable fragments and complete representations of algebras*, J. Symbolic Logic **73** (2008), no. 1, 65–89. MR 2387933 (2008m:03130)
- [3] Claude Berge, *Graphs and hypergraphs*, North-Holland Publishing Co., Amsterdam, 1973, Translated from the French by Edward Minieka, North-Holland Mathematical Library, Vol. 6. MR 0357172 (50 #9640)
- [4] Stephen D. Comer, *Color schemes forbidding monochrome triangles*, Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1983), vol. 39, 1983, pp. 231–236. MR 734546 (85f:05052)
- [5] ———, *Combinatorial aspects of relations*, Algebra Universalis **18** (1984), no. 1, 77–94.
- [6] M. F. Frias and R. D. Maddux, *Non-embeddable simple relation algebras*, Algebra Universalis **38** (1997), no. 2, 115–135. MR 1608984 (99a:03063)
- [7] Leon Henkin, J. Donald Monk, and Alfred Tarski, *Cylindric algebras. Part I. With an introductory chapter: General theory of algebras*, North-Holland Publishing Co., Amsterdam, 1971, Studies in Logic and the Foundations of Mathematics, Vol. 64. MR 0314620 (47 #3171)
- [8] ———, *Cylindric algebras. Part II*, Studies in Logic and the Foundations of Mathematics, vol. 115, North-Holland Publishing Co., Amsterdam, 1985. MR 781930 (86m:03095b)
- [9] Robin Hirsch, *Completely representable relation algebras*, Bull. IGPL **3** (1995), no. 1, 77–91. MR 1330986 (96d:03082)
- [10] Robin Hirsch and Ian Hodkinson, *Complete representations in algebraic logic*, J. Symbolic Logic **62** (1997), no. 3, 816–847. MR 1472125 (98m:03123)

- [11] ———, *Relation Algebras by Games*, Studies in Logic and the Foundations of Mathematics, vol. 147, North-Holland Publishing Co., Amsterdam, 2002, With a foreword by Wilfrid Hodges. MR 1935083 (2003m:03001)
- [12] ———, *Strongly representable atom structures of relation algebras*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1819–1831. MR 1887031 (2002k:03113)
- [13] ———, *Strongly representable atom structures of cylindric algebras*, J. Symbolic Logic **74** (2009), no. 3, 811–828. MR 2548463 (2011c:03147)
- [14] Robin Hirsch, Ian Hodkinson, and Roger D. Maddux, *Provability with finitely many variables*, Bull. Symbolic Logic **8** (2002), no. 3, 348–379. MR 1931348 (2003m:03013)
- [15] Ian Hodkinson, *Atom structures of cylindric algebras and relation algebras*, Ann. Pure Appl. Logic **89** (1997), no. 2-3, 117–148. MR 1490103 (99c:03103)
- [16] Ian Hodkinson and Yde Venema, *Canonical varieties with no canonical axiomatisation*, Trans. Amer. Math. Soc. **357** (2005), no. 11, 4579–4605. MR 2156722 (2006e:03106)
- [17] Mohamed Khaled and Tarek Sayed Ahmed, *Classes of algebras that are not closed under completions*, Bull. Sect. Logic Univ. Łódź **38** (2009), no. 1-2, 29–43. MR 2605423 (2011a:03071)
- [18] ———, *On complete representations of algebras of logic*, Log. J. IGPL **17** (2009), no. 3, 267–272. MR 2507303 (2011a:03072)
- [19] Roger C. Lyndon, *The representation of relational algebras*, Ann. of Math. (2) **51** (1950), 707–729. MR 0037278 (12,237a)
- [20] Roger Maddux, *Some varieties containing relation algebras*, Trans. Amer. Math. Soc. **272** (1982), no. 2, 501–526. MR 662049 (84a:03079)
- [21] Roger D. Maddux, *Topics in relation algebras*, Ph.D. thesis, University of California, Berkeley, 1978, pp. iii+241.
- [22] ———, *Finite integral relation algebras*, Universal Algebra and Lattice Theory (Charleston, S.C., 1984), Springer, Berlin, 1985, pp. 175–197.
- [23] ———, *Nonfinite axiomatizability results for cylindric and relation algebras*, J. Symbolic Logic **54** (1989), no. 3, 951–974. MR 1011183 (90f:03099)
- [24] ———, *Relation Algebras*, Studies in Logic and the Foundations of Mathematics, vol. 150, Elsevier B. V., Amsterdam, 2006. MR 2269199 (2007j:03096)
- [25] J. Donald Monk, *Nonfinitizability of classes of representable cylindric algebras*, J. Symbolic Logic **34** (1969), 331–343. MR 0256861 (41 #1517)
- [26] ———, *Completions of Boolean algebras with operators*, Math. Nachr. **46** (1970), 47–55. MR 0277369 (43 #3102)
- [27] Tarek Sayed Ahmed, *Neat embedding is not sufficient for complete representability*, Bull. Sect. Logic Univ. Łódź **36** (2007), no. 1-2, 29–35. MR 2357188 (2008g:03103)
- [28] ———, *A note on atom structures of relation and cylindric algebras*, Int. J. Algebra **2** (2008), no. 9-12, 595–601. MR 2443834 (2009g:03101)
- [29] ———, *A note on atom structures of relation and cylindric algebras*, Bull. Sect. Logic Univ. Łódź **37** (2008), no. 1, 29–35. MR 2441102 (2009g:03100)
- [30] ———, *Weakly representable atom structures that are not strongly representable, with an application to first order logic*, MLQ Math. Log. Q. **54** (2008), no. 3, 294–306. MR 2417802 (2009d:03161)
- [31] ———, *A simple construction of representable relation algebras with non-representable completions*, MLQ Math. Log. Q. **55** (2009), no. 3, 237–244. MR 2519240 (2010i:03072)
- [32] Tarek Sayed Ahmed and Basim Samir, *The class $\mathbf{SNr}_3\mathbf{CA}_k$ is not closed under completions*, Log. J. IGPL **16** (2008), no. 5, 427–429. MR 2453362
- [33] Alfred Tarski, *Contributions to the theory of models. III*, Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 56–64 = Indagationes Math. 17, 56–64 (1955). MR 0066303 (16,554h)

APPENDIX A. A representation result

The next theorem was invoked in the proof of Theorem 2.

Theorem 4. *Assume \mathbf{A} is an atomic symmetric integral relation algebra containing a flexible trio. Then \mathbf{A} has the 1-point extension property and $\mathbf{A} \in \mathbf{RRA}$.*

Proof. Once it has been shown that \mathbf{A} has the 1-point extension property, it follows from some additional observations about the behavior of identity elements that the

set $B_k(\mathbf{A})$ of basic k -by- k matrices of atoms of \mathbf{A} is a relational basis for \mathbf{A} whenever $k \geq 3$ (see Def. 9). Then $\mathbf{A} \in \mathbf{RA}_k$ for all $k \geq 3$ because \mathbf{A} is atomic and has a k -dimensional relational basis, hence $\mathbf{A} \in \bigcap_{k \geq 3} \mathbf{RA}_k = \mathbf{RRA}$. In fact, when \mathbf{A} is atomic (as in the proof of Theorem 1) it is easy to prove directly from the 1-point extension property that \mathbf{A} has a complete representation on ω .

We show next that there is a function f such that for any diversity atoms x and y , we have $f(x, y) \in \{a, b, c\}$ and

$$(21) \quad x; f(x, y) = y; f(x, y) \geq 0'$$

For an arbitrary diversity atom x , consider the set

$$(22) \quad Z_x := \{z : x; z \geq 0', 0' \geq z \in \text{At}(\mathbf{A})\}.$$

If $x \in \{a, b, c\}$ then $a, b, c \in Z_x$ by (12). If $x \notin \{a, b, c\}$ then by (13), $\{a, b, c\} \cap Z_x$ has at least two elements. Consequently, if y is another, possibly different, diversity atom of \mathbf{A} , then, since Z_x and Z_y are subsets of the 3-element set $\{a, b, c\}$ and they each contain at least two elements, they must intersect. We choose a value in the intersection as $f(x, y)$. There are several ways to do this. We pick this one—

- (A) if $a \in Z_x \cap Z_y$ then $f(x, y) = a$,
- (B) if $a \notin Z_x \cap Z_y$ and $b \in Z_x \cap Z_y$ then $f(x, y) = b$,
- (C) if $a \notin Z_x \cap Z_y$ and $b \notin Z_x \cap Z_y$ then $f(x, y) = c$.

It is obvious from (22) that (21) holds in the first two cases. We need to show (21) also holds in the third case (C), *i.e.*, that under the assumptions $a \notin Z_x \cap Z_y$ and $b \notin Z_x \cap Z_y$ we have $c \in Z_x \cap Z_y$. But if $c \notin Z_x \cap Z_y$ then we would conclude that $Z_x \cap Z_y$ is empty, since it is a subset of $\{a, b, c\}$ that excludes each of a , b , and c by our assumptions, contrary to the observations made above.

For the 1-point extension property, assume $k \in \omega$, $\mu \in B_k(\mathbf{A})$, $\mu_{l,m}$ satisfies the identity condition, x, y are diversity atoms of \mathbf{A} , and $\mu_{i,j} \leq x; y$ for some fixed $i, j < k$. We will prove that μ has a 1-point extension $\mu' \in B_{k+1}(\mathbf{A})$ such that $\mu \subseteq \mu'$, $\mu'_{i,k} = x = \mu'_{k,i}$, $\mu'_{k,j} = y = \mu'_{j,k}$, and if $k > l \neq i, j$ then

$$\mu'_{l,k} = \mu'_{k,l} = f(x, y)$$

Note that by definitions and (21) we have

$$(23) \quad x; \mu'_{k,l} = 0' = y; \mu'_{k,l}.$$

Having chosen $\mu'_{k,l}$ to be either a or b or c , we must check for each $l < k$ whether the first two crucial cycle equations below hold, and finally whether the third equation holds for those points $l, m < k$ where $l \neq m$ and $\{l, m\} \cap \{i, j\} = \emptyset$.

$$\begin{array}{ll} \mu_{i,l} \leq \mu'_{i,k}; \mu'_{k,l} & \text{i.e., } [\mu'_{i,k}, \mu'_{k,l}, \mu_{i,l}] \text{ is a cycle} \\ \mu_{j,l} \leq \mu'_{j,k}; \mu'_{k,l} & \text{i.e., } [\mu'_{j,k}, \mu'_{k,l}, \mu_{j,l}] \text{ is a cycle} \\ \mu_{l,m} \leq \mu'_{l,k}; \mu'_{k,m} & \text{i.e., } [\mu'_{l,k}, \mu'_{k,m}, \mu_{l,m}] \text{ is a cycle} \end{array}$$

The first two equations hold by (23) (their right sides are $0'$). For the third equation, first note that $\mu'_{l,k} = \mu'_{k,m}$ because the value depends only on x, y , not on l or m . The right side of the third equation is therefore $a; a$ or $b; b$ or $c; c$, but $a; a = b; b = c; c = 1$, so the third equation holds. \square

APPENDIX B. **A** non-representation result

Theorem 5. *Assume*

- (i) $\mathbf{E} \subseteq \mathbf{A} \in \mathbf{NA}$,
- (ii) \mathbf{E} is finite and symmetric, $1' \in \text{At}(\mathbf{E})$, and \mathbf{E} has $p > 3$ atoms,
- (iii) \mathbf{E} has no 1-cycles: $u; u \cdot u = 0$ if $0' \geq u \in \text{At}(\mathbf{E})$,
- (iv) \mathbf{A} is finite and symmetric, $1' \in \text{At}(\mathbf{A})$, and some diversity atom of \mathbf{E} is the join of at least p^{p-1} atoms of \mathbf{A} .

Then

- (i) $\mathbf{A} \notin \mathbf{SRaCA}_{p+1}$,
- (ii) $\mathbf{Ca}(B_3(\mathbf{A})) \notin \mathbf{SNr}_3\mathbf{CA}_{p+1}$.

Proof of (i). Let the atoms of \mathbf{E} be $1' = \mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{p-1}$, where $p \geq 3$ and \mathbf{a}_1 is a diversity atom of \mathbf{E} which is the join of at least p^{p-1} atoms of \mathbf{A} . Let $\mathbf{c}(x)$ be the atom of \mathbf{E} containing $x \in \text{At}(\mathbf{A})$. We refer to $\mathbf{c}(x)$ as the “color” of x (or “cover”, as in the definition of splitting).

Assume, for the sake of obtaining a contradiction, that $\mathbf{A} \subseteq \mathbf{Ra}(\mathbf{D})$ for some $\mathbf{D} \in \mathbf{CA}_{p+1}$. All the elements of \mathbf{A} , in particular all the atoms, are 2-dimensional elements of \mathbf{D} , i.e.,

$$(24) \quad \text{At}(\mathbf{A}) \subseteq \text{Nr}_2\mathbf{D}.$$

If $1 < q \leq p+1$ and $x \in D$, we say x is **q -color-ordered** if $\mathbf{c}(u) = \mathbf{c}(v)$ whenever $u, v \in \text{At}(\mathbf{A})$, $0 \leq i < j < k < q$, and $x \leq \mathbf{s}_i^0 \mathbf{s}_j^1 u \cdot \mathbf{s}_i^0 \mathbf{s}_k^1 v$.

The element $x \in D$ is **q -covered** if there are atoms $u_{i,j} \in \text{At}(\mathbf{A})$ for $0 \leq i < j < q$ such that $x \leq \prod_{0 \leq i < j < q} \mathbf{s}_i^0 \mathbf{s}_j^1 u_{i,j}$, in which case the atoms $u_{i,j}$ are said to be a **q -covering** of x .

The atoms in a q -covering of a non-zero $x \in D$ are unique, for if there are further atoms $v_{i,j} \in \text{At}(\mathbf{A})$, $0 \leq i < j < q$, such that $x \leq \prod_{0 \leq i < j < q} \mathbf{s}_i^0 \mathbf{s}_j^1 v_{i,j}$, then, since substitution is a complete Boolean endomorphism by [7, 1.5.3], we have

$$\begin{aligned} 0 \neq x &\leq \prod_{0 \leq i < j < q} \mathbf{s}_i^0 \mathbf{s}_j^1 u_{i,j} \cdot \prod_{0 \leq i < j < q} \mathbf{s}_i^0 \mathbf{s}_j^1 v_{i,j} \\ &= \prod_{0 \leq i < j < q} \mathbf{s}_i^0 \mathbf{s}_j^1 (u_{i,j} \cdot v_{i,j}), \end{aligned}$$

but if $u_{i,j} \neq v_{i,j}$ then, since distinct atoms are disjoint, a zero occurs with a contradiction ensuing. Thus $u_{i,j} = v_{i,j}$ whenever $0 \leq i < j < q$.

We will construct by induction for each dimension from $q = 2$ up to $q = p+1$ a set $S_q \subseteq \text{Nr}_q\mathbf{D}$ such that

- (i) S_q has at least p^{p+1-q} elements.
- (ii) Every $x \in S_q$ is q -covered, q -color-ordered, and non-zero, and $x \leq \mathbf{s}_j^1 \mathbf{a}_1$ for $0 < j < q$.
- (iii) $\mathbf{c}_{q-1}x = \mathbf{c}_{q-1}y$ if $x, y \in S_q$.
- (iv) $\mathbf{c}(u) = \mathbf{c}(v)$ if $u, v \in \text{At}(\mathbf{A})$, $x, y \in S_q$, $x \leq \mathbf{s}_{q-2}^0 \mathbf{s}_{q-1}^1 u$, and $y \leq \mathbf{s}_{q-2}^0 \mathbf{s}_{q-1}^1 v$.
- (v) $u \neq v$ if $u, v \in \text{At}(\mathbf{A})$, $x, y \in S_q$, $x \leq \mathbf{s}_{q-1}^1 u$, $y \leq \mathbf{s}_{q-1}^1 v$, and $x \neq y$.
- (vi) $u \neq v$ if $0 < j < k < q$, $u, v \in \text{At}(\mathbf{A})$, $x \in S_q$, and $x \leq \mathbf{s}_j^1 u \cdot \mathbf{s}_k^1 v$.

Let $S_2 = \{x : \mathbf{a}_1 \geq x \in \text{At}(\mathbf{A})\}$.

Note that $S_2 \subseteq \text{Nr}_2\mathbf{D}$ by (24). Obviously S_2 has property (i) since there are at least p^{p-1} atoms below \mathbf{a}_1 . Let $x \in S_2$. Then x is 2-covered by itself (take $u_{0,1} = x$),

x is 2-color-ordered because the hypotheses in the definition of color-ordered are never met ($q = 2$ is too small), and x is not zero because it is an atom of \mathbf{A} . For the last part of property (ii), note that if $0 \leq i < j < q = 2$ then $j = 1$, and $x \leq \mathbf{a}_1$ by the definition of S_2 , so $x \leq \mathbf{a}_1 = \mathbf{s}_1^1 \mathbf{a}_1 = \mathbf{s}_j^1 \mathbf{a}_1$. Therefore S_2 has property (ii). Since \mathbf{A} is integral and $x \in S_2$ is non-zero, we have $x;1 = 1$, so

$$\begin{aligned}
 1 = x;1 &= c_2(s_2^1 x \cdot s_2^0 1) && \text{definition of ; in } \mathbf{Ra}(\mathbf{D}) \\
 &= c_2 s_2^1 x && [7, 1.5.3] \\
 &= c_1 s_1^2 x && [7, 1.5.9(i)] \\
 &= c_1 x && [7, 1.5.8(i)], c_2 x = x
 \end{aligned}$$

It follows that property (iii) holds for S_2 . For property (iv), note that since $q = 2$, $s_{q-2}^0 s_{q-1}^1$ is the identity mapping, hence the hypotheses are $u, v \in At(\mathbf{A})$, $x, y \in S_2$, $x \leq u$, and $y \leq v$, which imply $x = u$ and $y = v$ since u, v, x, y are atoms. We wish to show $c(u) = c(v)$, i.e., $c(x) = c(y)$, but this is true by the definition of S_2 . Since $q = 2$, the substitution s_{q-1}^1 is the identity mapping, hence the hypotheses of property (v) are $u, v \in At(\mathbf{A})$, $x, y \in S_q$, $x \leq u$, $y \leq v$, and $x \neq y$. But these hypotheses imply $u = x \neq y = v$, so the conclusion holds trivially. Thus S_2 has property (v). Finally, S_2 has property (vi) because the hypotheses cannot hold when $q = 2$.

Suppose we have a set S_q such that $q \geq 2$ and (i)–(vi). Choose an arbitrary but fixed $w \in S_q$, and let $S_q^w := S_q \sim \{w\}$. We will obtain a function h that sends every $x \in S_q^w$ to a $(q+1)$ -dimensional element $h(x) \in Nr_{q+1}\mathbf{D}$, and will choose S_{q+1} to be a subset of the range of h .

For every $x \in S_q^w$, we have

$$\begin{aligned}
 0 &\neq w && \text{(ii)} \\
 &= w \cdot c_{q-1} w && [7, 1.1.1(C_2)] \\
 &= w \cdot c_{q-1} x && \text{(iii)} \\
 &= w \cdot c_{q-1} s_{q-1}^q x && [7, 1.5.8(i)], c_q x = x \\
 &= w \cdot c_q s_q^{q-1} x && [7, 1.5.9(i)] \\
 &= c_q(w \cdot s_q^{q-1} x) && [7, 1.1.1(C_3)], c_q w = w \\
 &= c_q(w \cdot s_q^{q-1} x \cdot s_{q-1}^0 s_q^1(1)) && [7, 1.5.3] \\
 &= c_q \left(w \cdot s_q^{q-1} x \cdot s_{q-1}^0 s_q^1 \left(\sum_{y \in At(\mathbf{A})} y \right) \right) && \mathbf{A} \text{ is finite} \\
 &= \sum_{y \in At(\mathbf{A})} c_q(w \cdot s_q^{q-1} x \cdot s_{q-1}^0 s_q^1(y)) && [7, 1.5.3, 1.2.6]
 \end{aligned}$$

The distributive law holds in all Boolean algebras whenever all the joins and meets involved are finite, so

$$0 \neq w = \prod_{x \in S_q^w} \left(\sum_{y \in At(\mathbf{A})} c_q(w \cdot s_q^{q-1} x \cdot s_{q-1}^0 s_q^1(y)) \right)$$

$$= \sum_{f: S_q^w \rightarrow \text{At}(\mathbf{A})} \left(\prod_{x \in S_q^w} c_q(w \cdot s_q^{q-1} x \cdot s_{q-1}^0 s_q^1(f(x))) \right).$$

Consequently there must be some function $f : S_q^w \rightarrow \text{At}(\mathbf{A})$ such that

$$(25) \quad 0 \neq \prod_{x \in S_q^w} c_q(w \cdot s_q^{q-1} x \cdot s_{q-1}^0 s_q^1(f(x))).$$

Let f be such a function. From our chosen f we define additional functions $g, h : S_q^w \rightarrow D$ and an element $z \in D$ as follows.

$$(26) \quad g(x) = w \cdot s_q^{q-1} x \cdot s_{q-1}^0 s_q^1(f(x)) \quad \text{for all } x \in S_q^w$$

$$(27) \quad z = \prod_{x \in S_q^w} c_q(g(x))$$

$$(28) \quad h(x) = g(x) \cdot z \quad \text{for all } x \in S_q^w$$

Let $R = \{h(x) : x \in S_q^w\}$. We will show that R itself has properties (ii), (iii), (v), and (vi). Consequently every subset of R also has these properties. We will partition R into disjoint subsets that have property (iv) and prove that at least one of them must be large enough to also have property (i). We take S_{q+1} to be any such subset of R .

To see that R has property (iii), we observe that $c_q h(x) = c_q h(y)$ for all $x, y \in S_q^w$, because

$$\begin{aligned} c_q h(x) &= c_q(g(x) \cdot z) & (28) \\ &= c_q(g(x)) \cdot z & [7, 1.1.1(\text{C}_3)], c_q z = z \\ &= z & (27) \end{aligned}$$

It follows that $h(x) \neq 0$ for every $x \in S_q$, since $z \neq 0$ by (25). This is part of property (ii). For the last part of property (ii), we want to show $h(x) \leq s_j^1(a_1)$ whenever $0 < j < q+1$ and $x \in S_2$. We have $h(x) \leq g(x) \leq w \cdot s_q^{q-1} x$ by definitions (28) and (26), so there are two cases. First, assume $0 < j < q$. In this case we note that from $w \in S_q$ and (ii) for S_q we get $w \leq s_j^1 a_1$, so $h(x) \leq s_j^1 a_1$. Suppose $j = q$. In this case we have $x \leq s_j^1 a_1$ for $0 < j < q$ by (ii) for S_q since $x \in S_q$. In particular, $x \leq s_{q-1}^1 a_1$, so $h(x) \leq s_q^{q-1} s_{q-1}^1 a_1 = s_q^1 a_1$. We get the rest of property (ii) by showing $h(x)$ is $(q+1)$ -color-ordered and $(q+1)$ -covered for every $x \in S_q^w$. From $x \in S_q^w$ and property (ii) for S_q we know x is q -covered, so there are atoms $x_{i,j} \in \text{At}(\mathbf{A})$ such that

$$(29) \quad x \leq \prod_{0 \leq i < j < q} s_i^0 s_j^1(x_{i,j}).$$

Of course, we also know $w \in S_q$, so there is a q -covering $w_{i,j} \in \text{At}(\mathbf{A})$, $0 \leq i < j < q$, of w as well, where

$$(30) \quad w \leq \prod_{0 \leq i < j < q} s_i^0 s_j^1(w_{i,j}).$$

Let

$$(31) \quad t_{i,j} = \begin{cases} w_{i,j} & \text{if } 0 \leq i < j < q \\ x_{i,q-1} & \text{if } 0 \leq i < q-1 \text{ and } j = q \\ f(x) & \text{if } i = q-1 \text{ and } j = q \end{cases}$$

We shall see that $t_{i,j}$ is a $(q+1)$ -covering of $h(x)$. First, note that

$$(32) \quad s_q^{q-1}x \leq \prod_{0 \leq i < q-1} s_i^0 s_q^1(x_{i,q-1})$$

because if $0 \leq i < q-1$ then $x \leq s_i^0 s_{q-1}^1(x_{i,q-1})$ by (29), so

$$\begin{aligned} s_q^{q-1}x &\leq s_q^{q-1} s_i^0 s_{q-1}^1(x_{i,q-1}) & [7, 1.5.3] \\ &= s_i^0 s_q^{q-1} s_{q-1}^1(x_{i,q-1}) & [7, 1.5.10(\text{iii})] \\ &= s_i^0 s_q^1(x_{i,q-1}) & c_{q-1}x_{i,q-1} = x_{i,q-1}, [7, 1.5.11(\text{i})] \end{aligned}$$

Then we have

$$(33) \quad h(x) \leq g(x) = w \cdot s_q^{q-1}x \cdot s_{q-1}^0 s_q^1(f(x)) \quad (28), (26)$$

$$\leq \prod_{0 \leq i < j < q} s_i^0 s_j^1(w_{i,j}) \cdot \prod_{0 \leq i < q-1} s_i^0 s_q^1(x_{i,q-1}) \cdot s_{q-1}^0 s_q^1(f(x)) \quad (30), (32)$$

$$= \prod_{0 \leq i < j < q} s_i^0 s_j^1(t_{i,j}) \cdot \prod_{0 \leq i < q-1} s_i^0 s_q^1(t_{i,q}) \cdot s_{q-1}^0 s_q^1(t_{q-1,q}) \quad (31)$$

$$= \prod_{0 \leq i < j < q+1} s_i^0 s_j^1(t_{i,j})$$

so $h(x)$ is $(q+1)$ -covered.

To show $h(x)$ is $(q+1)$ -color-ordered, we assume $0 \leq i < j < k < q+1$ and must show $c(t_{i,j}) = c(t_{i,k})$. If $i < j < k < q$ then the first case in (31) applies to both $t_{i,j}$ and $t_{i,k}$, hence $t_{i,j} = w_{i,j}$ and $w_{i,k} = t_{i,k}$, but $c(w_{i,j}) = c(w_{i,k})$ because w is color-ordered, so we have $c(t_{i,j}) = c(t_{i,k})$. We may therefore assume $k = q$.

We need to observe before going on that if $q > 2$, then

$$\begin{aligned} w &\leq c_{q-1}w = c_{q-1}x && \text{property (iii) of } S_q \\ &= c_{q-1} \left(\prod_{0 \leq i < j < q-1} s_i^0 s_j^1(x_{i,j}) \right) && [7, 1.2.6], (29) \\ &= \prod_{0 \leq i < j < q-1} s_i^0 s_j^1(x_{i,j}) && c_{q-1}x_{i,j} = x_{i,j}, q-1 \geq 2 \end{aligned}$$

By the uniqueness of coverings this tells us that

$$(34) \quad w_{i,j} = x_{i,j} \text{ if } 0 \leq i < j < q-1.$$

If, in addition to $k = q$, we have $i < j < q-1$, then $q > 2$ and the first and second cases of (31) apply, so we have $t_{i,j} = w_{i,j}$ and $t_{i,k} = t_{i,q} = x_{i,q-1}$. But $w_{i,j} = x_{i,j}$ by (34). Also, x is color-ordered, so $c(x_{i,j}) = c(x_{i,q-1})$, which is equivalent to $c(t_{i,j}) = c(t_{i,k})$ by the previous equations.

The final case is that $i < j = q-1$ and $k = q$. The possibilities for i divide into two sub-cases, i is smaller than $q-2$, and i is equal to $q-2$. If $0 \leq i < q-2$ then $i < q-2 < q-1$, so $c(t_{i,j}) = c(w_{i,q-1}) = c(w_{i,q-2})$ since w is color-ordered by

property (ii) of S_q , and $\mathbf{c}(x_{i,q-2}) = \mathbf{c}(x_{i,q-1})$ since x is color-ordered, but $w_{i,q-2} = x_{i,q-2}$ by (34), so

$$\begin{aligned} \mathbf{c}(t_{i,j}) &= \mathbf{c}(w_{i,q-1}) & j = q-1, (31) \\ &= \mathbf{c}(w_{i,q-2}) & w \text{ is color-ordered} \\ &= \mathbf{c}(x_{i,q-2}) & (34) \\ &= \mathbf{c}(x_{i,q-1}) & x \text{ is color-ordered} \\ &= \mathbf{c}(t_{i,k}) & q = k, (31) \end{aligned}$$

We are reduced to assuming $i = q-2$, hence

$$\mathbf{c}(t_{i,j}) = \mathbf{c}(w_{q-2,q-1}) = \mathbf{c}(x_{q-2,q-1}) = \mathbf{c}(t_{i,k})$$

by (34) and the third case in (31).

We have shown that every $h(x)$ constructed from some $x \in S_q^w$ is non-zero, $(q+1)$ -covered, and $(q+1)$ -color-ordered. Thus R and all its subsets has property (ii).

To prove property (v) for R (and its subsets), we assume $x, y \in S_q^w$, $h(x) \neq h(y)$, $u, v \in \text{At}(\mathbf{A})$, $h(x) \leq \mathbf{s}_q^1 u$, $h(y) \leq \mathbf{s}_q^1 v$. We must show $u \neq v$. If we have a q -covering of x as in (29), then by (33) we get $u = x_{0,q-1}$ from $h(x) \leq \mathbf{s}_q^1 u$, and, similarly, $v = y_{0,q-1}$ from $h(y) \leq \mathbf{s}_q^1 v$ for some q -covering $y_{i,j}$ of y . Hence $x \leq \mathbf{s}_{q-1}^1(x_{0,q-1})$ and $y \leq \mathbf{s}_{q-1}^1(y_{0,q-1})$, so, by property (v) for S_q^w , we know $x_{0,q-1} \neq y_{0,q-1}$, i.e., $u \neq v$, as desired.

To prove property (vi) for R (and its subsets), we assume $0 < j < k < q+1$, $u, v \in \text{At}(\mathbf{A})$, $x \in S_q^w$, and $h(x) \leq \mathbf{s}_j^1 u \cdot \mathbf{s}_k^1 v$. If $k < q$, then $u = t_{0,j} = w_{0,j}$ and $v = t_{0,k} = w_{0,k}$ by (33) and (31), but $w \in S_q$, so by property (vi) for S_q , we have $w_{0,j} \neq w_{0,k}$, hence $u \neq v$. Suppose that $k = q$. In this case, by (33) and (31), we again have $u = t_{0,j} = w_{0,j}$ but this time $v = t_{0,q} = x_{0,q-1}$. Hence $w \leq \mathbf{s}_j^1 u$ and $x \leq \mathbf{s}_{q-1}^1 v$ by (29) and (30). If $j = q-1$ we note that $w \neq x$ since $x \in S_q^w$, hence $u \neq v$ by property (v) for S_q , which gives us $t_{0,j} \neq t_{0,q}$, i.e., $u \neq v$. If $j < q-1$ then $v = t_{0,q} = x_{0,q-1} \neq x_{0,j}$ for $0 < j < q-1$ by property (vi) for S_q , applied this time to x . But $x_{0,j} = w_{0,j} = t_{0,j}$ by (34) and (31), so again we have $t_{0,q} \neq t_{0,j}$.

We have proved R has properties (ii), (iii), (v), and (vi), and wish to show that h is one-to-one on S_q^w . Assume $x, y \in S_q^w$ and $x \neq y$. We want to show $h(x) \neq h(y)$. By property (ii) for S_q^w , x and y have q -coverings that include atoms $x_{0,q-1}, y_{0,q-1} \in \text{At}(\mathbf{A})$ satisfying $x \leq \mathbf{s}_{q-1}^1(x_{0,q-1})$ and $y \leq \mathbf{s}_{q-1}^1(y_{0,q-1})$. By (31) and (33) these last two equations imply $h(x) \leq \mathbf{s}_q^1(x_{0,q-1})$ and $h(y) \leq \mathbf{s}_q^1(y_{0,q-1})$. From $x \neq y$ we conclude by property (v) for S_q^w that $x_{0,q-1} \neq y_{0,q-1}$, which implies, by property (v) for R , that $h(x) \neq h(y)$, as desired.

Now we want to choose a subset S_{q+1} of R with property (iv) that contains at least $p^{p+1-(q+1)}$ elements. We partition R and let S_{q+1} be the largest piece. Recall from (33) that $h(x) \leq \mathbf{s}_{q-2}^0 \mathbf{s}_{q-1}^1(f(x))$ for every $x \in S_q^w$, and $f(x)$ has color $\mathbf{c}(f(x)) \in \text{At}(\mathbf{E})$. For every color \mathbf{a}_i we get a piece of R , namely

$$R_i := \{h(x) : x \in S_q^w, \mathbf{c}(f(x)) = \mathbf{a}_i\}.$$

Note that R is the disjoint union of the pieces, the number of pieces is p , and R has at least p^{p+1-q} elements because h is one-to-one and S_q^w has more than p^{p+1-q} elements. Consequently some piece has at least $p^{p+1-q}/p = p^{p-q}$ elements in it, and we let S_{q+1} be any such piece. Thus S_{q+1} has property (i). Every piece has

property (iv), so in particular S_{q+1} has this property. Finally, as a subset of R , S_{q+1} has all the other properties. This completes the construction of the sets S_q .

Consider what happens when $q = p + 1$. We may choose some $x \in S_{p+1}$ because S_{p+1} has at least one element, by property (i). Then x is $(p + 1)$ -covered, $(p + 1)$ -color-ordered, and non-zero by property (ii). Let x have $(p + 1)$ -covering $x_{i,j} \in At(\mathbf{A})$ for $0 \leq i < j < p + 1$.

Consider the set $\{c(x_{i,p}) : 0 \leq i < p\} \subseteq At(\mathbf{E})$. Note that $c(x_{0,p}) = \mathbf{a}_1 \neq 1'$ since $x_{0,p} \leq \mathbf{a}_1$ by property (ii). We can also show $c(x_{i,p}) \neq 1'$ for $0 < i < p$ because we have, by the covering of x , $x \leq s_i^1(x_{0,i}) \cdot s_i^0 s_p^1(x_{i,p}) \cdot s_p^1(x_{0,p})$ so it follows by [14, Lemma 10] that $[x_{0,i}, x_{i,p}, x_{0,p}]$ is a cycle, *i.e.*, $x_{0,i}; x_{i,p} \geq x_{0,p}$. If $c(x_{i,p}) = 1'$ then $x_{i,p} = 1'$ and we would get $x_{0,i} = x_{0,p}$, contradicting property (vi), which says $x_{0,i} \neq x_{0,p}$ for $0 < i < p$. Thus we know $c(x_{i,p})$ is a diversity atom of \mathbf{E} for $0 \leq i < p$.

The number of diversity atoms in \mathbf{E} is $p - 1$, but the size of the index set $\{i : 0 \leq i < p\}$ is p . Therefore some atom is repeated, *i.e.*, there are $0 \leq i < j < p$ such that $c(x_{i,p}) = c(x_{j,p})$. By the $(p + 1)$ -color-ordering of x , $c(x_{i,j}) = c(x_{i,p})$. Let $u = c(x_{i,j}) = c(x_{i,p}) = c(x_{j,p})$. We proved above that $u \neq 1'$. By the covering of x and property (ii) we have $0 \neq x \leq s_i^0 s_j^1(x_{i,j}) \cdot s_j^0 s_p^1(x_{j,p}) \cdot s_i^0 s_p^1(x_{i,p})$, hence by the definition of u and [14, Lemma 10] we have $0 \neq u; u \cdot u$. Since $u \neq 1'$, this contradicts the assumption that \mathbf{E} has no such diversity atom as the u we have found. \square

Proof of (ii). Assume to the contrary that $\mathbf{Ca}(B_3(\mathbf{A})) \in \mathbf{SNr}_3 \mathbf{CA}_{p+1}$. Then

$$\begin{aligned} \mathbf{A} &\cong \mathbf{Ra}(\mathbf{Ca}(B_3(\mathbf{A}))) \in \mathbf{Ra}^* \mathbf{SNr}_3 \mathbf{CA}_{p+1} \\ &= \mathbf{SRa}^* \mathbf{Nr}_3 \mathbf{CA}_{p+1} && [8, 5.3.13] \\ &= \mathbf{SRa}^* \mathbf{CA}_{p+1} && [8], \text{ defs} \end{aligned}$$

contradicting part (i) that says $\mathbf{A} \notin \mathbf{SRa} \mathbf{CA}_{p+1}$. \square

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